1 Hey, man. Whazzup (inside the light-cone)?

In the last lecture we have seen that

$$\Delta(x-y) = [\phi(x), \phi(y)] = \int d^3\tilde{p}\, \left[e^{-ip(x-y)} - e^{ip(x-y)}\right] . \tag{1.1}$$

By noticing that this is a Lorentz-invariant 3-momentum integral (thanks to the measure $d^3\tilde{p}$), we then figured out that

$$[\phi(t, \boldsymbol{x}), \phi(t, \boldsymbol{y})] = 0\,, \tag{1.2}$$

implies that

$$\Delta(x-y) = 0\,, \tag{1.3}$$

if x and y are space-like separated, *i.e.*, $(x-y)^2 < 0$. This shows that our quantized scalar theory is causal.

But what happens for $\Delta(x-y)$ for time-like separations? *E.g.*, taking $x = (t, 0, 0, 0)$ and $y = (0, 0, 0, 0)$ gives $[\phi(x), \phi(y)] \propto \exp(-imt) - \exp(imt)$, where the $\exp(-imt)$ factor arises from first term in (1.1) as follows

$$\begin{aligned} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\boldsymbol{p}}}\, e^{-i\omega_{\boldsymbol{p}} t} &= \frac{4\pi}{(2\pi)^3} \int_0^\infty dp\, \frac{p^2}{2\sqrt{p^2+m^2}}\, e^{-it\sqrt{p^2+m^2}} \\ &= \frac{1}{4\pi^2} \int_m^\infty dE\, \sqrt{E^2-m^2}\, e^{-iEt} \\ &= \frac{m}{8\pi t} \left[Y_1(mt) + i J_1(mt)\right] \underset{t\to\infty}{\propto} e^{-imt} . \end{aligned} \tag{1.4}$$

Here the second line is obtained by simply changing variables $p \to (E^2 - m^2)^{1/2}$. In order to arrive at the final answer, one only needs to know that the Bessel functions of first and second kind, $J_1(x)$ and $Y_1(x)$, behave like $J_1(x)/x \propto \sin x - \cos x$ and $Y_1(x)/x \propto \sin x + \cos x$ in the relevant limit $x \to \infty$. An analog calculation gives the $\exp(imt)$ term.

The causal structure of the real Klein-Gordon theory can also be probed in a different way. Let's create a particle at the space-time point y. What is the amplitude to find it at point x? This question can be answered by looking at

$$D(x-y) = \langle 0\, |\phi(x)\phi(y)|\, 0\rangle = \int d^3\tilde{p}\, e^{-ip(x-y)} . \tag{1.5}$$

The function $D(x-y)$ is called *propagator* and is a Lorentz-invariant 3-momentum integral.

Let us now evaluate (1.5) for purely space-like separations, *i.e.*, $x - y = (0, \boldsymbol{r})$.[1] The propagator is then

$$\begin{aligned} D(x-y) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\boldsymbol{p}}}\, e^{i\boldsymbol{p}\cdot\boldsymbol{r}} = \frac{2\pi}{(2\pi)^3} \int_0^\infty dp\, \frac{p^2}{2\omega_{\boldsymbol{p}}} \frac{e^{ipr} - e^{-ipr}}{ipr} \\ &= \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^\infty dp\, \frac{p\, e^{ipr}}{\sqrt{p^2+m^2}} . \end{aligned} \tag{1.6}$$

[1]Notice that for purely time-like separations one would obtain the result (1.4).

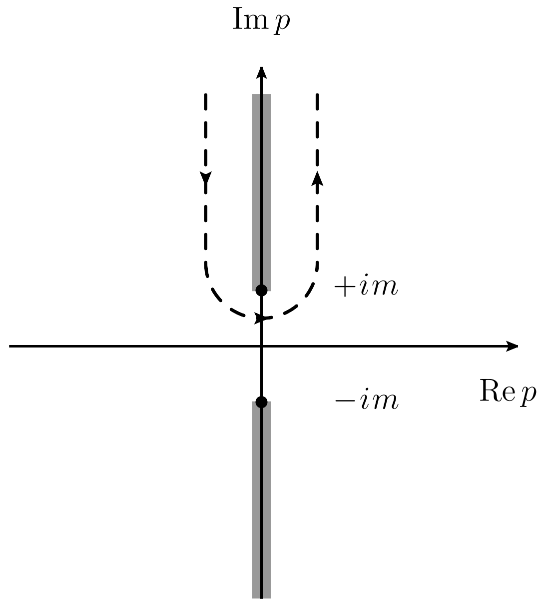

Figure 1.1: Branch cuts of the propagator $D(x-y)$ for a space-like transition.

Here we have first introduced spherical coordinates, then performed the integration over the azimuthal and polar angles, and finally changed variables in the second term from $p \to -p$ in order to combine the result into one term. The integrand in (1.6), considered as a complex function of p, has branch cuts on the imaginary axis starting at $\pm im$. In order to evaluate the integral we push the contour up to wrap around the upper branch cut. The chosen integration contour is shown in Figure 1.1. Defining $\rho = -ip$, we then recast (1.6) into

$$D(x-y) = \frac{1}{4\pi^2 r}\int_m^\infty d\rho\,\frac{\rho e^{-\rho r}}{\sqrt{\rho^2-m^2}} = \frac{1}{4\pi^2 r}\,mK_1(mr) \underset{r\to\infty}{\propto} e^{-mr}\,, \tag{1.7}$$

where the modified Bessel function $K_1(x)$ scales like $K_1(x) = \big(\sqrt{\pi/(2x)}+\mathcal{O}(x^{-3/2})\big)\,e^{-x}$ in the limit of $x \to \infty$. The latter equation tells us that the propagator $\Delta(x-y)$ decays exponentially quickly outside the light-cone but, nonetheless, it is non-vanishing. The quantum field appears to leak out of the causal region. Yet, we have learnt in the lecture that space-like measurements commute and the theory is causal. How do we reconcile these two facts?

The puzzle is readily resolved by realizing that the relation (1.1), expressed in terms of propagators, takes the form

$$\Delta(x-y) = [\phi(x),\phi(y)] = D(x-y) - D(y-x) = 0\,. \tag{1.8}$$

What is the physical meaning of this result? It simply means that for $(x-y)^2 < 0$, there is no Lorentz-invariant way to order events. If a particle can travel in a space-like direction from x to y, it can just as easily travel from y to x.[2] In any measurement, the amplitudes for these two possible events cancel, so that the underlying QFT is causal.

[2] When $x-y$ is space-like, a continuous Lorentz transformation can take $x-y$ to $y-x$.