The forward-backward asymmetry in electron-positron annihilation

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Introduction: Electroweak precision physics
I.: Higher order corrections
II: Infrared-safe definition of the observable
III: Outline of the calculation
IV.: Results
Our current paradigm: The Standard Model

The Higgs boson: The Standard Model predicts a scalar particle, which gives rise to the mass of all other particles.
- yet to be discovered -

Up to the Higgs boson manifests itself only through quantum corrections!

(Electroweak Working Group, hep-ex/0509008.)
Electroweak precision physics

Precision observables allow us to extract the values of the five input parameters for the Standard model at the $Z$-pole.

Input parameters are:\n\[
\alpha(m_Z^2), \alpha_s(m_Z^2), m_Z, m_t, m_H.\]

Check how individual measurements agree with the results of this fit.

The forward-backward asymmetry for $b$-quarks shows the largest pull.

(Electroweak Working Group, hep-ex/0509008.)
The forward-backward asymmetry

\[ A_{FB} = \frac{N_F - N_B}{N_F + N_B} \]

But: Free quarks are not observed, instead hadronic jets are seen in the detector!
Perturbation theory

Due to the smallness of the coupling constants $\alpha$ and $\alpha_s$, we may compute observables at high energies reliable in perturbation theory,

$$\langle O \rangle = \langle O \rangle_{LO} + \frac{\alpha_s}{2\pi} \langle O \rangle_{NLO} + \left(\frac{\alpha_s}{2\pi}\right)^2 \langle O \rangle_{NNLO} + \ldots$$

provided that the observable is infrared-safe!

In particular, it is required that they do not change value, if infinitessimal soft or collinear particles are added.

$$O_{n+1}(p_1, \ldots, p_{n+l}) \rightarrow O_n(p_1', \ldots, p_n'),$$

The forward-backward asymmetry is measured experimentally with a precision at the per cent level.

To match this precision the inclusion of QCD corrections in a theoretical calculation is mandatory.
Calculation of the **NNLO QCD corrections** to the forward-backward asymmetry in massless QCD:

\[
A_{FB} = A_{FB}^{(0)} \left( 1 + \frac{\alpha_s}{2\pi} B_{FB} + \left( \frac{\alpha_s}{2\pi} \right)^2 C_{FB} \right) + O\left( \alpha_s^3 \right),
\]

- G. Altarelli and B. Lampe, 1993;
- V. Ravindran and W. L. van Neerven, 1998;

**NLO corrections including mass corrections:**


Partial results for **mass corrections at NNLO:**

W. Bernreuther, A. Brandenburg, and P. Uwer, 2000; W. Bernreuther et al., 2006;
Some examples of diagrams contributing to the various orders in perturbation theory:

**LO:**

**NLO:**

**NNLO:**

Purely virtual diagrams cancel in the correction to the asymmetry!
Definitions used in the literature

How to define the direction of the $b$-quark in the presence of additional partons?

- Define the direction by the momentum of the quark.
- Use the thrust axis as direction.

How to treat the $bb\bar{b}\bar{b}$ final state if two $b$-quarks are tagged?

- Count it once.
- Count it twice.

The experimental analysis seems to have used the thrust axis and counted $bb\bar{b}\bar{b}$ final states with weight two.
Catani ans Seymour have shown, that none of the combinations thrust axis/ quark axis and weight two/ weight one yields an infrared finite observable.

The divergence is proportional to

$$\int_{0}^{1} dz P_{q \to q\bar{q}}(z) \ln \frac{Q^2}{m_b^2}$$

To absorb this divergence one can introduce a $b$-quark fragmentation function. This brings along additional uncertainties related to non-perturbative physics.
Questions

Can the introduction of the fragmentation function and dependence on non-perturbative physics be avoided?

How to define the forward-backward asymmetry in an infrared-safe way?

What about a jet axis?
Jet algorithms

The most fine-grained look at hadronic events consistent with infrared safety is given by classifying the particles into jets.

Ingredients:

- a resolution variable $y_{ij}$ where a smaller $y_{ij}$ means that particles $i$ and $j$ are “closer”;
- a combination procedure which combines two four-momenta into one;
- a cut-off $y_{min}$ which provides a stopping point for the algorithm.

A typical algorithm:

- for each pair $i, j$, calculate $y_{ij}$
- select pair with smallest $y_{ij}$; if $y_{ij} < y_{min}$, combine $i$ and $j$
- repeat until the smallest $y_{ij} > y_{min}$
Example: The Durham or $k_\perp$-algorithm for partons, whose flavour is not detected. (Dokshitzer, 1991)

Resolution variable:

$$y_{ij}^{DURHAM} = \frac{2(1 - \cos \theta_{ij})}{Q^2} \min(E_i^2, E_j^2)$$

Combination procedure:

$$p_{(ij)}^\mu = p_i^\mu + p_j^\mu.$$
The Durham algorithm is not infrared-safe for jets with flavour, since at order $\alpha_s^2$ a soft gluon can split into a soft $q\bar{q}$ pair.

The Durham measure

$$y_{ij}^{DURHAM} = \frac{2(1 - \cos \theta_{ij})}{Q^2} \min(E_i^2, E_j^2)$$

assumes that parton emission has a soft and a collinear divergence.

However, there is no soft divergence in the $g \rightarrow q\bar{q}$ splitting.
The flavour-$k_\perp$ algorithm

In order to account for tagged flavours modify the Durham measure

$$y_{ij}^{DURHAM} = \frac{2(1 - \cos \theta_{ij})}{Q^2} \min(E_i^2, E_j^2)$$

towards

$$y_{ij}^{flavour} = \frac{2(1 - \cos \theta_{ij})}{Q^2} \times \begin{cases} 
\min(E_i^2, E_j^2), & \text{softer of } i, j \text{ is flavourless,} \\
\max(E_i^2, E_j^2), & \text{softer of } i, j \text{ is flavoured.}
\end{cases}$$

This yields an infrared-safe definition of jets if flavours are tagged.

Definition of the forward-backward asymmetry

- Assign **flavour number** $+1$ to a $b$-quark and $-1$ to a $\bar{b}$-quark. All other particles have flavour number zero.

- **Cluster particles into jets**, using the flavour-$k_\perp$ algorithm.

- If two particles are combined, the **flavour numbers are added**.

- **Select two jet events**, where one jet has flavour number $> 0$.

- The **jet axis** of this jet **defines** the **direction** relevant to the forward-backward asymmetry.
Calculation of the NLO and NNLO corrections

To compute for this definition the NLO and NNLO corrections, a general purpose program for NNLO corrections to $e^+e^- \rightarrow 2\text{ jets}$ is used.

S.W., 2006.

The relevant matrix elements are known for a long time.


**Difficulty:** Cancellation of IR divergences.
General methods at NLO

Fully differential NLO Monte Carlo programs need a general method to handle the cancelation of infrared divergencies.

- **Phase space slicing**

- **Subtraction method**
The dipole formalism

The dipole formalism is based on the subtraction method. The NLO cross section is rewritten as

\[ \sigma^{NLO} = \int_{n+1} d\sigma^R + \int_n d\sigma^V \]

\[ = \int_{n+1} (d\sigma^R - d\sigma^A) + \int_n \left( d\sigma^V + \int_1 d\sigma^A \right) \]

The approximation \( d\sigma^A \) has to fulfill the following requirements:

- \( d\sigma^A \) must be a proper approximation of \( d\sigma^R \) such as to have the same pointwise singular behaviour in \( D \) dimensions as \( d\sigma^R \) itself. Thus, \( d\sigma^A \) acts as a local counterterm for \( d\sigma^R \) and one can safely perform the limit \( \varepsilon \to 0 \).

- Analytic integrability in \( D \) dimensions over the one-parton subspace leading to soft and collinear divergences.
The subtraction method at NNLO

- **Singular behaviour**
  - Factorization of *tree amplitudes* in *double unresolved limits*, Berends, Giele, Cambell, Glover, Catani, Grazzini, Del Duca, Frizzo, Maltoni, Kosower ‘99
  - Factorization of *one-loop amplitudes* in *single unresolved limits*, Bern, Del Duca, Kilgore, Schmidt, Kosower, Uwer, Catani, Grazzini, ‘99

- **Extension of the subtraction method to NNLO**  Kosower; S.W.; Anastasiou, Melnikov, Petriello; Kilgore; Gehrmann-De Ridder, Gehrmann, Glover, Heinrich; Frixione, Grazzini; Somogyi, Trócsányi and Del Duca;

- **Applications:**
  - $pp \rightarrow W$, Anastasiou, Dixon, Melnikov, Petriello ‘03,
  - $e^+e^- \rightarrow 2\text{ jets}$, Anastasiou, Melnikov, Petriello ‘04,
The subtraction method at NNLO

Contributions at NNLO:

\[
\begin{align*}
  d\sigma_{n+2}^{(0)} &= \left( A_{n+2}^{(0)} \ast A_{n+2}^{(0)} \right) d\phi_{n+2}, \\
  d\sigma_{n+1}^{(1)} &= \left( A_{n+1}^{(0)} \ast A_{n+1}^{(1)} + A_{n+1}^{(1)} \ast A_{n+1}^{(0)} \right) d\phi_{n+1}, \\
  d\sigma_{n}^{(2)} &= \left( A_{n}^{(0)} \ast A_{n}^{(2)} + A_{n}^{(2)} \ast A_{n}^{(0)} + A_{n}^{(1)} \ast A_{n}^{(1)} \right) d\phi_{n},
\end{align*}
\]

Adding and subtracting:

\[
\begin{align*}
  \langle O \rangle_{n}^{\text{NNLO}} &= \int \left( O_{n+2} d\sigma_{n+2}^{(0)} - O_{n+1} \circ d\alpha_{n+1}^{(0,1)} - O_{n} \circ d\alpha_{n}^{(0,2)} \right) \\
  &+ \int \left( O_{n+1} d\sigma_{n+1}^{(1)} + O_{n+1} \circ d\alpha_{n+1}^{(0,1)} - O_{n} \circ d\alpha_{n}^{(1,1)} \right) \\
  &+ \int \left( O_{n} d\sigma_{n}^{(2)} + O_{n} \circ d\alpha_{n}^{(0,2)} + O_{n} \circ d\alpha_{n}^{(1,1)} \right).
\end{align*}
\]
NNLO subtraction terms

The \((n + 2)\)-parton contribution:

\[
\int \left( \sigma_{n+2}^{(0)} - o_{n+1} \circ \alpha^{(0,1)}_{n+1} - o_n \circ \alpha^{(0,2)}_n \right), \quad d\alpha^{(0,2)}_n = d\alpha^{(0,2)}_{(0,0)n} - d\alpha^{(0,2)}_{(0,1)n}.
\]

has to be integrable for all double and single unresolved limits.

The \((n + 1)\)-parton contribution:

\[
\int \left( \sigma_{n+1}^{(1)} + o_{n+1} \circ \alpha^{(0,1)}_{n+1} - o_n \circ \alpha^{(1,1)}_n \right), \quad d\alpha^{(1,1)}_n = d\alpha^{(1,1)}_{(1,0)n} + d\alpha^{(1,1)}_{(0,1)n}
\]

has to be integrable over single unresolved limits.

In addition, explicit poles in \(\varepsilon\) have to cancel.
Example: $qgg\bar{q}$ final state for $e^+e^- \rightarrow 2$ jets

NNLO subtraction terms for the $(n+2)$-parton configuration:

$$d\alpha_{(0,0)}^{(0,2)} = \frac{1}{2} \left\{ \frac{N}{2} C_F \left[ A_4^0(1, 2, 3, 4) + A_4^0(1, 3, 2, 4) \right] - \frac{1}{2N} C_F \left[ A_{4,sc}^0(1, 2, 3, 4) + A_{4,sc}^0(1, 3, 2, 4) \right] \right\} \left| a_2^{(0)} \right|^2$$

$$d\alpha_{(0,1)}^{(0,2)} = \frac{1}{2} \left\{ \frac{N}{2} \left[ D_3^0(1, 2, 3) + D_3^0(1, 3, 2) + D_3^0(4, 2, 3) + D_3^0(4, 3, 2) \right] - \frac{1}{2N} \left[ A_3^0(1, 2, 4) + A_3^0(1, 3, 4) \right] \right\} C_F A_3^0(1', 2', 3') \left| a_2^{(0)} \right|^2.$$
Spin-averaged antenna functions

Spin-averaged $qgg\bar{q}$ antenna function obtained from the matrix element $\gamma^* \rightarrow qgg\bar{q}$:

$$A_4^{(0)}(q_1, g_2, g_3, \bar{q}_4) = e g^2 \left[ (T^2 T^3)_{14} A_4^{(0)}(q_1, g_2, g_3, \bar{q}_4) + (T^3 T^2)_{14} A_4^{(0)}(q_1, g_3, g_2, \bar{q}_4) \right]$$

$$\left| A_4^{(0)} \right|^2 = e^2 g^4 N(N^2 - 1) / 4 \left( A_4^{(0)}(2, 3), A_4^{(0)}(3, 2) \right) \left( \begin{array}{cc} 1 - \frac{1}{N^2} & -\frac{1}{N^2} \\ \frac{1}{N^2} & 1 - \frac{1}{N^2} \end{array} \right) \left( \begin{array}{c} A_4^{(0)}(2, 3) \\ A_4^{(0)}(3, 2) \end{array} \right)$$

Leading-colour antenna function:

$$A_4^0(1, 2, 3, 4) = \left| A_4^{(0)}(2, 3) \right|^2 / \left| A_2^{(0)} \right|^2$$

Subleading-colour:

$$A_{4,sc}^0(1, 2, 3, 4) + A_{4,sc}^0(1, 3, 2, 4) = \left| A_4^{(0)}(2, 3) + A_4^{(0)}(3, 2) \right|^2 / \left| A_2^{(0)} \right|^2$$
\[ A_4^0(1, 2, 3, 4) = \]
\[
\frac{1}{48 s_{1234}} \left( \frac{48 s_{1234}}{s_{234}^2} + \frac{32 s_{1234}}{s_{23}^2} + \frac{48 s_{1234}}{s_{123}^2} + \frac{48 s_{23} - 48 s_{123} + 64 s_{1234}}{s_{12} s_{234}} + \frac{-32 s_{123} s_{1234} + 16 s_{123}^2 - 32 s_{34} s_{1234} + 16 s_{34}^2 + 32 s_{1234}^2}{s_{12} s_{23} s_{234}} \right) \\
- \frac{-48 s_{12} - 96 s_{23} - 48 s_{34} - 96 s_{1234}}{s_{123} s_{234}} - \frac{16 s_{1234}}{s_{34} s_{234}} - \frac{-32 s_{123} s_{1234} + 16 s_{123}^2 - 32 s_{1234} s_{234} + 16 s_{234}^2 + 32 s_{1234}^2}{s_{12} s_{23} s_{34}} + \frac{96}{s_{123}} + \frac{32 s_{1234}}{s_{12} s_{34}} \\
- \frac{16 s_{1234}}{s_{12} s_{123}} + \frac{64 s_{12} s_{1234} s_{1234}}{s_{23} s_{234}^2 s_{123}} + \frac{16 s_{1234}}{s_{23} s_{234}^2} + \frac{64 s_{12} s_{1234} - 32 s_{12}^2 + 64 s_{34} s_{1234} - 32 s_{34}^2 - 128 s_{1234}^2}{s_{23} s_{123} s_{234}} + \frac{16 s_{23} s_{1234}}{s_{34} s_{234}^2} + \frac{48 s_{23} - 48 s_{234} + 64 s_{1234}}{s_{123} s_{34}} \\
+ \frac{48 s_{12} - 48 s_{123} + 32 s_{1234}}{s_{23} s_{234}} + \frac{64 s_{12} s_{1234}}{s_{23} s_{234}^2} + \frac{64 s_{34} s_{1234}}{s_{23} s_{123}^2} - \frac{64 s_{34} s_{1234}}{s_{23}^2 s_{234}} + \frac{48 s_{34} - 48 s_{234} + 32 s_{1234}}{s_{23} s_{123}} + \frac{32 s_{34}^2 s_{1234}}{s_{23}^2 s_{234}^2} - \frac{64 s_{12} s_{1234}}{s_{23} s_{234}} \\
+ \frac{32 s_{12}^2 s_{1234}}{s_{23}^2 s_{123}^2} + \frac{16 s_{12} s_{1234}}{s_{12} s_{123}^2} + \frac{-32 s_{12} s_{1234} + 16 s_{12}^2 - 32 s_{1234} s_{234} + 16 s_{234}^2 + 32 s_{1234}^2}{s_{23} s_{123} s_{34}} + \frac{96}{s_{123} s_{34}} \\
- \frac{-32 s_{12} s_{1234} - 16 s_{23}^2 + 48 s_{34} s_{1234} - 16 s_{34}^2 - 64 s_{1234}^2}{s_{12} s_{123} s_{234}} + \frac{48 s_{12} s_{1234} - 16 s_{12}^2 - 32 s_{23} s_{1234} - 16 s_{23}^2 - 64 s_{1234}^2}{s_{123} s_{34} s_{234}} \\
+ \frac{32 s_{12} s_{1234}^2 + 16 s_{23}^2 s_{1234} + 32 s_{1234}^3}{s_{12} s_{123} s_{34} s_{234}} + \frac{-32 s_{12} s_{1234} + 16 s_{123} s_{1234} - 32 s_{1234}^2}{s_{12} s_{34} s_{234}} + \frac{-32 s_{12} s_{1234} + 16 s_{1234} s_{234} - 32 s_{1234}^2}{s_{12} s_{123} s_{34}} + \frac{96}{s_{234}} \right) \]
In the collinear limit spin correlations remain:

\[ A_\mu \frac{k_\perp \cdot k_\perp}{k_\perp^2} A_\nu, \]

where \( k_\perp = (1 - z)p_i + zp_j - (1 - 2z)\frac{y}{1 - y}p_k. \)

Let \( \phi \) be the azimuthal angle of \( p_i \) around \( p_i + p_j \). Then

\[ A_\mu \frac{k_\perp \cdot k_\perp}{k_\perp^2} A_\nu \sim C_0 + C_2 \cos(2\phi + \alpha). \]

One can perform the average with two points:

\[ \phi, \quad \phi + \frac{\pi}{2}, \]

while all other coordinates remain fixed.
Phase space generation

Dimension of phase space for $n$ final state particles: $3n - 4$.

Split the phase space into different channels, according to which invariants are the smallest.

For each channel, use a parameterization such that $\varphi$ is along a coordinate axis:

$$d\phi_{n+1} = d\phi_n \ d\phi_{\text{dipole}},$$

$$d\phi_{\text{dipole}} = \frac{s_{ijk}}{32\pi^3} (1 - y) \ dy \ dz \ d\varphi.$$  

Construct the momenta of the $(n + 1)$ event from the ones of the $n$ parton event and the values of $y$, $z$ and $\varphi$. 
Numerical results for the forward-backward asymmetry of $b$-quarks

Perturbative expansion:

\[
A_{FB} = A_{FB}^{(0)} \left( 1 + \frac{\alpha_s}{2\pi} B_{FB} + \left( \frac{\alpha_s}{2\pi} \right)^2 C_{FB} \right) + O(\alpha_s^3),
\]

Select two-jet events defined by the flavour-$k_\perp$ algorithm and a given $y_{cut}$.

Leading order result independent of $y_{cut}$:

\[
A_{FB,b}^{(0)} = 0.11161.
\]

QCD corrections:

<table>
<thead>
<tr>
<th>$y_{cut}$</th>
<th>$B_{FB,b}$</th>
<th>$C_{FB,b}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>$-0.070 \pm 0.005$</td>
<td>$-0.4 \pm 0.8$</td>
</tr>
<tr>
<td>0.03</td>
<td>$-0.145 \pm 0.003$</td>
<td>$-1.7 \pm 0.5$</td>
</tr>
<tr>
<td>0.1</td>
<td>$-0.294 \pm 0.002$</td>
<td>$-4.3 \pm 0.3$</td>
</tr>
<tr>
<td>0.3</td>
<td>$-0.512 \pm 0.001$</td>
<td>$-10.2 \pm 0.1$</td>
</tr>
<tr>
<td>0.9</td>
<td>$-0.565 \pm 0.001$</td>
<td>$-13.4 \pm 0.1$</td>
</tr>
</tbody>
</table>
Plot

LO(x)

"fb_b_NLO.dat"

"fb_b_NNLO.dat"

thrust_NLO(x)

quark_NLO(x)
Numerical results for the forward-backward asymmetry of $c$-quarks

Perturbative expansion:

$$A_{FB} = A_{FB}^{(0)} \left( 1 + \frac{\alpha_s}{2\pi} B_{FB} + \left( \frac{\alpha_s}{2\pi} \right)^2 C_{FB} \right) + O(\alpha_s^3),$$

Select two-jet events defined by the flavour-$k_\perp$ algorithm and a given $y_{cut}$. Leading order result independent of $y_{cut}$:

$$A_{FB,c}^{(0)} = 0.08003.$$

QCD corrections:

<table>
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</table>
Summary

- The forward-backward asymmetry shows the largest discrepancy in a fit of the Standard Model parameter.
- Experimental analysis based on an infrared-unsafe definition.
- Infrared-safe definition of the forward-backward asymmetry.
- Calculation of the NLO and NNLO QCD corrections.
- The corrections are small, useful observable also for a future linear collider.