

Introduction to theoretical elementary particle physics

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1 Overview

Literature:

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- M. Böhm, A. Denner and H. Joos, Gauge Theories of the Strong and Electroweak Interactions, Teubner, 2001.
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1.1 Units

Convention:

$$\hbar = c = 1$$

For example, the equation

$$E^2 - c^2 \vec{p}^2 = m^2 c^4$$

simplifies to

$$E^2 - \vec{p}^2 = m^2.$$

Energy is measured in eV:

$$1 \text{ eV} = 1.6021764 \cdot 10^{-19} J$$

$$1 \text{ keV} = 10^3 \text{ eV}$$

$$1 \text{ MeV} = 10^6 \text{ eV}$$

$$1 \text{ GeV} = 10^9 \text{ eV}$$

$$1 \text{ TeV} = 10^{12} \text{ eV}$$

Momenta are measured in $\text{eV}/c = \text{eV}$, masses are given in $\text{eV}/c^2 = \text{eV}$.

From the uncertainty relation

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$$

it follows that lengths are given in $\hbar c \cdot \text{eV}^{-1} = \text{eV}^{-1}$.

Cross sections are given in barn:

$$1 \text{ barn} = 10^{-28} \text{ m}^2$$

$$1 \text{ nbarn} = 10^{-9} \text{ barn}$$

$$1 \text{ pbarn} = 10^{-12} \text{ barn}$$

$$1 \text{ fbarn} = 10^{-15} \text{ barn}$$

Conversion constant:

$$(\hbar c)^2 = 0.389379292 \cdot 10^9 \text{ GeV}^2 \text{ pbarn}$$

1.2 The fundamental forces

We know four fundamental forces: the strong force, the weak force, the electro-magnetic force and the gravitational force. Particle physics deals with the strong, the weak and the electro-magnetic force. The gravitational force is negligible against the other three forces at present energy scales.

The standard model is based on a local gauge theory with gauge group

$$SU(3) \otimes SU(2) \otimes U(1)$$

$SU(3)$ corresponds to the strong interactions, $SU(2)$ to the weak isospin, $U(1)$ to the hypercharge. The symmetry of the subgroup $SU(2) \otimes U(1)$ is spontaneously broken down to the familiar $U(1)_{el-magn}$ symmetry of electro-magnetic interactions.

1.3 The elementary particles

The spin of the particles:

- **Fermions** have half-integer spin. In the standard model all fermions have spin 1/2. In extensions of the standard model higher spins may occur, e.g. the gravitino with spin 3/2.
- **Bosons** have integer spin. All bosons, which have been observed up to now, have spin 1. In addition the standard model predicts a boson of spin 0, the Higgs boson. In extensions of the standard model there might be particles of higher spin, e.g. a graviton of spin 2.

1.3.1 Spin 1/2 particles

Quarks: Quarks feel the strong, the weak and the electro-magnetic forces. There are six quarks:

up, $Q_u = \frac{2}{3}$	charm, $Q_c = \frac{2}{3}$	top, $Q_t = \frac{2}{3}$
$m_u < 10 \text{ MeV}$	$m_c = 1.15 - 1.35 \text{ GeV}$	$m_t = 174 \pm 5 \text{ GeV}$
down, $Q_d = -\frac{1}{3}$	strange, $Q_s = -\frac{1}{3}$	bottom, $Q_b = -\frac{1}{3}$
$m_d < 10 \text{ MeV}$	$m_s = 80 - 130 \text{ MeV}$	$m_b^{\overline{\text{MS}}} = 4.1 - 4.4 \text{ GeV}$ $m_b^{1\text{S}} = 4.6 - 4.9 \text{ GeV}$

The different quark types (up, down, strange, charm, bottom, top) are called “**flavours**”.

Leptons: Leptons do not feel the strong interaction. There are six leptons:

$\nu_e, Q_{\nu_e} = 0$	$\nu_\mu, Q_{\nu_\mu} = 0$	$\nu_\tau, Q_{\nu_\tau} = 0$
$m_{\nu_e} < 3 \text{ eV}$	$m_{\nu_\mu} < 0.19 \text{ MeV}$	$m_{\nu_\tau} < 18.2 \text{ MeV}$
$e, Q_e = -1$	$\mu, Q_\mu = -1$	$\tau, Q_\tau = -1$
$m_e = 511 \text{ keV}$	$m_\mu = 105.7 \text{ MeV}$	$m_\tau = 1.78 \text{ GeV}$

Neutrinos are electrically neutral and interact only through the weak force.

The family structure of the standard model: The fermions can be grouped into three families:

$$\begin{pmatrix} u \\ d \\ \nu_e \\ e \end{pmatrix}, \begin{pmatrix} c \\ s \\ \nu_\mu \\ \mu \end{pmatrix}, \begin{pmatrix} t \\ b \\ \nu_\tau \\ \tau \end{pmatrix}.$$

The families differ only by the masses of their members.

1.3.2 Spin 1 particles

Within the standard model the mediators of the interactions are spin 1 particles.

The strong interaction: SU(3): The gauge group $SU(3)$ describes the strong interaction. The number of generators for a group $SU(N)$ is $N^2 - 1$, therefore there are 8 generators for $SU(3)$, and hence 8 gauge bosons for the strong interactions. The gauge bosons of the strong interaction are called **gluons**. The fields are denoted by

$$A_\mu^a,$$

where a runs from 1 to 8.

The weak isospin: SU(2): The weak interaction is described by the gauge group $SU(2)$. There are three generators

$$W_\mu^1, W_\mu^2, W_\mu^3,$$

each with two polarization states. After electro-weak symmetry breaking we have

$$W_\mu^+, W_\mu^-, Z_\mu,$$

with three polarization states. We have the following relations:

$$\begin{aligned} W_\mu^\pm &= \frac{1}{\sqrt{2}} (W_\mu^1 \mp iW_\mu^2), \\ Z_\mu &= -\sin\theta_W B_\mu + \cos\theta_W W_\mu^3. \end{aligned}$$

The hypercharge: U(1): The last piece of gauge-symmetries within the standard model is given by an abelian $U(1)$ gauge symmetry, the hypercharge. The field is denoted by

$$B_\mu$$

After electro-weak symmetry breaking the photon field A_μ is given as a linear combination of B_μ and W_μ^3 :

$$A_\mu = \cos\theta_W B_\mu + \sin\theta_W W_\mu^3.$$

Note that

$$\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} B_\mu \\ W_\mu^3 \end{pmatrix}$$

and that A_μ remains a massless field.

Quantum numbers of the fermions in the electro-weak sector: The left-handed components (u_L, d_L) and (ν_L, e_L) transform as the fundamental representation under the $SU(2)$ group. The right-handed components u_R, d_R, ν_R and e_R transform as a singlet under the $SU(2)$ group.

In detail one has:

	I_3	Y	Q		I_3	Y	Q
u_L	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	u_R	0	$\frac{4}{3}$	$\frac{2}{3}$
d_L	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{3}$	d_R	0	$-\frac{2}{3}$	$-\frac{1}{3}$
ν_L	$\frac{1}{2}$	-1	0	ν_R	0	0	0
e_L	$-\frac{1}{2}$	-1	-1	e_R	0	-2	-1

The electric charge is given by the Gell-Mann-Nishijima formula:

$$Q = I_3 + \frac{Y}{2}$$

Remark: The table contains a right-handed neutrino, which does not interact with any other particle.

1.3.3 Spin 0 particles

Up to now, no elementary spin 0 particle has been observed.

The Higgs boson: Within the standard model one assumes an additional complex scalar field, transforming as the fundamental representation of $SU(2)$:

$$\phi(x) = \begin{pmatrix} \phi^+(x) \\ \frac{1}{\sqrt{2}}(v + H(x) + i\chi(x)) \end{pmatrix}.$$

$\phi^+(x)$ is a complex field (two real components). The three components $\phi(x)$ and $\chi(x)$ are absorbed as the longitudinal modes of W_μ^\pm and Z_μ . $H(x)$ is the Higgs field.

1.4 Experiments

The first experiments were fixed-target experiments (**deep inelastic scattering** of electrons on proton targets).

Accelerators:

- LEP, e^+e^- , 210 GeV, $L = 10^{32} \text{ cm}^{-2} \text{ s}^{-1}$;
- TEVATRON, $p\bar{p}$, 1.96 TeV, $L = 5 \cdot 10^{30} \text{ cm}^{-2} \text{ s}^{-1}$;
- HERA, e^-p , $e^- : 30 \text{ GeV}$, $p : 960 \text{ GeV}$, $L = 75 \cdot 10^{30} \text{ cm}^{-2} \text{ s}^{-1}$;
- LHC, pp , 14 TeV, $L = 10^{34} \text{ cm}^{-2} \text{ s}^{-1}$;
- Linear Collider, e^+e^- , 500 GeV, $L = 5 \cdot 10^{34} \text{ cm}^{-2} \text{ s}^{-1}$;

Quarks and gluons are not directly observed in these experiments. Instead one observes **hadronic jets**. A jet is a bunch of particles moving in the same direction. Particles in a jet are not necessarily elementary.

1.5 Observed, but not elementary particles

Due to confinement, quarks and gluons cannot be observed as free particles. In experiments we observe particles which are colour-singlets like **mesons** and **baryons**. Within the quark model, mesons are $q\bar{q}$ -states and baryons qqq -states. Mesons and baryons are called collectively **hadrons**.

Examples are:

Mesons: Pions, kaons, η 's, D -mesons, J/ψ , ...

Baryons: protons, neutrons, Σ , Ξ , ...

2 Review of quantum mechanics

2.1 The harmonic oscillator in classical mechanics

The Lagrange function for the harmonic oscillator reads

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2.$$

From the Euler-Lagrange equation

$$\frac{d}{dt} \frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} = 0$$

follows the equation of motion

$$\ddot{x} + \omega^2 x = 0,$$

which has the solution

$$x(t) = Ae^{i\omega t} + Be^{-i\omega t}.$$

Remark: The conjugate momentum is given by

$$p = \frac{\delta L}{\delta \dot{x}} = m\dot{x},$$

and the Hamilton function reads

$$H = p\dot{x} - L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2 x^2 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2.$$

2.2 The harmonic oscillator in quantum mechanics

In quantum mechanics the harmonic oscillator is described by a wave function $\psi(x, t)$. This wave function can be expanded into an orthonormal basis. We denote by $|x, t\rangle$ a wave function, which is an eigenvector of the position operator \hat{x} :

$$\hat{x}|x, t\rangle = x|x, t\rangle.$$

We are interested in the transition amplitude

$$\langle x_f, t_f | x_i, t_i \rangle,$$

which gives us the probability that the system which was in the eigenstate $|x_i, t_i\rangle$ at time t_i will be found in the state $|x_f, t_f\rangle$ at time t_f .

2.2.1 Creation and annihilation operators

The time evolution of the wave function is given by the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \psi(x,t) = \hat{H} \psi(x,t),$$

where the Hamilton operator is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2.$$

We make the Ansatz

$$\psi(x,t) = \hat{U}(t, t_i) |x, t_i\rangle$$

The evolution operator satisfies the equation

$$i\hbar \frac{\partial}{\partial t} \hat{U}(t, t_i) = \hat{H} \hat{U}(t, t_i).$$

If the Hamilton operator is time-independent, the solution for $\hat{U}(t, t_i)$ is given by

$$\hat{U}(t, t_i) = \exp\left(-\frac{i}{\hbar}(t-t_i) \cdot \hat{H}\right).$$

Remark: If the Hamilton operator \hat{H} depends on the time t , a formal solution for $\hat{U}(t, t_i)$ is given by

$$\hat{U}(t, t_i) = T \exp\left(-\frac{i}{\hbar} \int_{t_i}^t dt' \hat{H}(t')\right).$$

Here, T denotes the time-ordering operator, which orders operators from right to left in non-decreasing time. Expanding the exponential one obtains

$$\begin{aligned} \hat{U}(t, t_i) &= 1 - \frac{i}{\hbar} \int_{t_i}^t dt_1 \hat{H}(t_1) + \left(\frac{i}{\hbar}\right)^2 \int_{t_i}^t dt_1 \hat{H}(t_1) \int_{t_i}^{t_1} dt_2 \hat{H}(t_2) \\ &\quad - \left(\frac{i}{\hbar}\right)^3 \int_{t_i}^t dt_1 \hat{H}(t_1) \int_{t_i}^{t_1} dt_2 \hat{H}(t_2) \int_{t_i}^{t_2} dt_3 \hat{H}(t_3) + \dots \end{aligned}$$

Note that the factor $1/n!$ disappears.

To determine $\hat{U}(t, t_i) |x, t_i\rangle$ we expand $|x, t_i\rangle$ into eigenstate of the Hamilton operator. These eigenstates will be labelled $|n\rangle$:

$$\hat{U}(t, t_i) |x, t_i\rangle = \exp\left(-\frac{i}{\hbar}(t-t_i) \cdot \hat{H}\right) |x, t_i\rangle = \sum_n e^{-\frac{i}{\hbar}(t-t_i)E_n} |n\rangle \langle n|x, t_i\rangle.$$

For the energy eigenstates we have

$$\hat{H}|n\rangle = E_n|n\rangle.$$

To find these eigenstates we define two operators

$$a = \frac{\omega m \hat{x} + i \hat{p}}{\sqrt{2\omega m \hbar}}, \quad a^\dagger = \frac{\omega m \hat{x} - i \hat{p}}{\sqrt{2\omega m \hbar}}.$$

If we introduce the characteristic length

$$x_0 = \sqrt{\frac{\hbar}{\omega m}}$$

we can equally write them as

$$a = \frac{1}{\sqrt{2}} \left(\frac{x}{x_0} + x_0 \frac{d}{dx} \right), \quad a^\dagger = \frac{1}{\sqrt{2}} \left(\frac{x}{x_0} - x_0 \frac{d}{dx} \right).$$

From

$$[\hat{x}, \hat{p}] = \left[\hat{x}, \frac{\hbar}{i} \frac{d}{dx} \right] = i\hbar$$

it follows that

$$[a, a^\dagger] = 1.$$

The Hamilton operator can be rewritten as

$$\hat{H} = \frac{1}{2} \hbar \omega (a^\dagger a + a a^\dagger) = \hbar \omega \left(a^\dagger a + \frac{1}{2} \right).$$

We call

$$\hat{n} = a^\dagger a$$

the number operator and the problem of finding the energy eigenstates is reduced to the problem of finding the eigenstates of the number operator. We have

$$n \langle n|n\rangle = \langle n|a^\dagger a n\rangle = \langle a n|a n\rangle \geq 0.$$

Therefore $n \geq 0$ and the lowest energy state corresponds to $n = 0$. Since the norm of $a|0\rangle$ vanishes, we have

$$\begin{aligned} a|0\rangle &= 0, \\ \left(\frac{d}{dx} + \frac{x}{x_0^2} \right) |0\rangle &= 0. \end{aligned}$$

A solution is given by

$$|0\rangle = (\sqrt{\pi} x_0)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \left(\frac{x}{x_0}\right)^2\right).$$

One easily shows that

- $a^\dagger|n\rangle$ is an eigenstate with eigenvalue $n + 1$.
- $a|n\rangle$ is an eigenstate with eigenvalue $n - 1$.

a^\dagger is called a creation operator, a is called an annihilation operator.

Therefore one finds

$$|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle = (2^n n! \sqrt{\pi} x_0)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \left(\frac{x}{x_0}\right)^2\right) H_n\left(\frac{x}{x_0}\right),$$

where $H_n(t)$ are the Hermite polynomials.

The corresponding energies are given by

$$E_n = \hbar\omega \left(n + \frac{1}{2}\right).$$

Finally, we get

$$\langle x_f, t_f | x_i, t_i \rangle = \sum_n \langle x_f, t_f | n \rangle e^{-\frac{i}{\hbar}(t-t_i)E_n} \langle n | x_i, t_i \rangle.$$

2.2.2 Path integrals

An alternative approach to determine the transition amplitude $\langle x_f, t_f | x_i, t_i \rangle$ divides the time interval $(t_f - t_i)$ into $n + 1$ small sub-intervals with time steps at

$$t_i, t_1, t_2, \dots, t_n, t_f.$$

At each intermediate time step we insert a complete set of states

$$\int_{-\infty}^{\infty} dx |x, t_j\rangle \langle x, t_j| = 1.$$

Therefore

$$\langle x_f, t_f | x_i, t_i \rangle = \int_{-\infty}^{\infty} dx_n \dots \int_{-\infty}^{\infty} dx_1 \langle x_f, t_f | x_n, t_n \rangle \langle x_n, t_n | x_{n-1}, t_{n-1} \rangle \dots \langle x_1, t_1 | x_i, t_i \rangle$$

Let us study $\langle x_{j+1}, t_{j+1} | x_j, t_j \rangle$. If the time interval $(t_{j+1} - t_j)$ is small, we have

$$\langle x_{j+1}, t_{j+1} | x_j, t_j \rangle = \langle x_{j+1} | e^{-\frac{i}{\hbar}(t_{j+1}-t_j)\hat{H}} | x_j \rangle \approx \langle x_{j+1} | 1 - \frac{i}{\hbar}(t_{j+1} - t_j)\hat{H} | x_j \rangle$$

We have

$$\langle x_{j+1}|x_j \rangle = \delta(x_{j+1} - x_j) = \frac{1}{\hbar} \int_{-\infty}^{\infty} \frac{dp_j}{2\pi} \exp\left(\frac{i}{\hbar} p_j (x_{j+1} - x_j)\right)$$

Remark:

$$\delta(x-y) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip(x-y)}$$

Therefore

$$\begin{aligned} & \langle x_{j+1} | 1 - \frac{i}{\hbar} (t_{j+1} - t_j) \hat{H} | x_j \rangle \\ & \approx \langle x_{j+1} | x_j \rangle - \frac{i}{\hbar} (t_{j+1} - t_j) \left\langle x_{j+1} \left| -\frac{\hbar^2}{2m} \frac{d^2}{dx_j^2} + \frac{1}{2} m \omega^2 x_j^2 \right| x_j \right\rangle \\ & = \frac{1}{\hbar} \int_{-\infty}^{\infty} \frac{dp_j}{2\pi} \exp\left(\frac{i}{\hbar} p_j (x_{j+1} - x_j)\right) \left(1 - \frac{i}{\hbar} (t_{j+1} - t_j) \left(\frac{p_j^2}{2m} + \frac{1}{2} m \omega^2 x_j^2\right)\right) \\ & = \frac{1}{\hbar} \int_{-\infty}^{\infty} \frac{dp_j}{2\pi} \exp\left(\frac{i}{\hbar} p_j (x_{j+1} - x_j)\right) \left(1 - \frac{i}{\hbar} (t_{j+1} - t_j) H(x_j, p_j)\right) \\ & \approx \frac{1}{\hbar} \int_{-\infty}^{\infty} \frac{dp_j}{2\pi} \exp\left(\frac{i}{\hbar} p_j (x_{j+1} - x_j) - \frac{i}{\hbar} (t_{j+1} - t_j) H(x_j, p_j)\right) \\ & \approx \frac{1}{\hbar} \int_{-\infty}^{\infty} \frac{dp_j}{2\pi} \exp\left(\frac{i}{\hbar} (t_{j+1} - t_j) (p_j \dot{x}_j - H(x_j, p_j))\right) \end{aligned}$$

Let us set

$$\Delta T = i(t_{j+1} - t_j)$$

With the help of

$$\int_{-\infty}^{\infty} \frac{dy}{2\pi} e^{-\frac{1}{2}\alpha y^2 + wy} = \frac{1}{\sqrt{2\pi\alpha}} e^{\frac{w^2}{2\alpha}}$$

we may perform the integration over p_j :

$$\int_{-\infty}^{\infty} \frac{dp_j}{2\pi} \exp\left(\frac{\Delta T}{\hbar} \left(p_j \dot{x}_j - \frac{p_j^2}{2m}\right)\right) = \sqrt{\frac{\hbar m}{2\pi\Delta T}} \exp\left(\frac{\Delta T}{\hbar} \frac{1}{2} m \dot{x}_j^2\right)$$

Therefore

$$\begin{aligned}\langle x_{j+1} | 1 - \frac{i}{\hbar}(t_{j+1} - t_j)\hat{H} | x_j \rangle &\approx \sqrt{\frac{m}{2\pi\hbar\Delta T}} \exp\left(\frac{\Delta T}{\hbar} \left(\frac{1}{2}m\dot{x}_j^2 - \frac{1}{2}m\omega^2 x_j^2\right)\right) \\ &= \sqrt{\frac{m}{2\pi\hbar\Delta T}} \exp\left(\frac{\Delta T}{\hbar} L(x_j, \dot{x}_j)\right)\end{aligned}$$

Finally we get

$$\begin{aligned}\langle x_f, t_f | x_i, t_i \rangle &= \int_{-\infty}^{\infty} dx_n \dots \int_{-\infty}^{\infty} dx_1 \langle x_f, t_f | x_n, t_n \rangle \langle x_n, t_n | x_{n-1}, t_{n-1} \rangle \dots \langle x_1, t_1 | x_i, t_i \rangle \\ &= \left(\frac{m}{2\pi\hbar\Delta T}\right)^{\frac{n+1}{2}} \int_{-\infty}^{\infty} dx_n \dots \int_{-\infty}^{\infty} dx_1 \prod_{j=0}^n \exp\left(\frac{i}{\hbar}(t_{j+1} - t_j)L(x_j, \dot{x}_j)\right),\end{aligned}$$

with $t_0 = t_i$. We rewrite this as

$$\langle x_f, t_f | x_i, t_i \rangle \sim \int \mathcal{D}x(t) \exp\left(\frac{i}{\hbar} \int_{t_i}^{t_f} dt L(x(t), \dot{x}(t))\right) = \int \mathcal{D}x(t) \exp\left(\frac{i}{\hbar} S\right)$$

2.3 Summary

The quantum mechanical harmonic oscillator shows already several concepts, which will reappear later in quantum field theory:

- Annihilation and creation operator.
- Transition amplitudes can be expressed as path integrals.
- The appearance of the Lagrange function and the action in the path integral.

Summary of the last lecture

Transition amplitude with creation and annihilation operators:

$$\langle x_f, t_f | x_i, t_i \rangle = \sum_n \langle x_f, t_f | n \rangle e^{-\frac{i}{\hbar}(t-t_i)E_n} \langle n | x_i, t_i \rangle.$$

Eigenstates of the Hamilton operator:

$$|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle$$

Creation and annihilation operator:

$$a = \frac{1}{\sqrt{2}} \left(\frac{x}{x_0} + x_0 \frac{d}{dx} \right), \quad a^\dagger = \frac{1}{\sqrt{2}} \left(\frac{x}{x_0} - x_0 \frac{d}{dx} \right).$$

The canonical commutation relation:

$$[a, a^\dagger] = 1.$$

The transition amplitude with path integrals:

$$\langle x_f, t_f | x_i, t_i \rangle \sim \int \mathcal{D}x(t) \exp \left(\frac{i}{\hbar} \int_{t_i}^{t_f} dt L(x(t), \dot{x}(t)) \right) = \int \mathcal{D}x(t) \exp \left(\frac{i}{\hbar} S \right)$$

3 Review of special relativity

3.1 Four-vectors and the metric

Four-vectors: The space-time coordinates (ct, x, y, z) are regarded as the components of a vector in a four-dimensional space.

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z.$$

$$\begin{aligned} x^\mu &= (x^0, x^1, x^2, x^3), \\ &= (x^0, \vec{x}). \end{aligned}$$

Greek indices μ, ν, \dots , which take the values 0, 1, 2, 3, are used to denote the components of a four-vector. Latin indices i, j, \dots are used to denote the (spatial) components of a three-vector. They take the values 1, 2, 3.

The distance between two points in four-dimensional space-time is

$$s_{ab} = (x_a^0 - x_b^0)^2 - (x_a^1 - x_b^1)^2 - (x_a^2 - x_b^2)^2 - (x_a^3 - x_b^3)^2.$$

$s_{ab}^2 > 0$ time-like distance;
there exists a frame, in which events a and b occur in the same place.

$s_{ab}^2 < 0$ space-like distance;
there exists a frame, in which events a and b occur at the same time.

$s_{ab}^2 = 0$ lighth-like distance;
light cone

Two events can only be related by causality, if the distance between them is ≥ 0 . This follows directly from the finiteness of the speed of light.

We define the metric tensor $g_{\mu\nu}$ by

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The distance is then given by

$$s_{ab} = \sum_{\mu=0}^3 \sum_{\nu=0}^3 g_{\mu\nu} (x_a^\mu - x_b^\mu) (x_a^\nu - x_b^\nu).$$

Summation convention of Einstein: The symbol of the sum is dropped and it is understood, that there is an implicit summation over any pair of indices, which occurs twice. Within a pair, one index has to be an upper index, the other one a lower index. Therefore:

$$s_{ab} = g_{\mu\nu} (x_a - x_b)^\mu (x_a - x_b)^\nu.$$

We call a four-vector x^μ with an upper index a contravariant four-vector, and we call a four-vector x_μ with a lower index a covariant four-vector. The relation between the two is given by

$$x_\mu = g_{\mu\nu} x^\nu.$$

Therefore we can write the distance equally as

$$s_{ab} = (x_a - x_b)_\mu (x_a - x_b)^\mu = (x_a - x_b)^\mu (x_a - x_b)_\mu.$$

Remark: The geometry defined by the quadratic form $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ is non-Euclidean. The special case of a four-dimensional space with metric $\text{diag}(1, -1, -1, -1)$ is often called Minkowski space.

3.2 The Lorentz group

Axioms for a group: Let G be a non-empty set with a binary operation. G is called a group, if it satisfies the axioms

- Associativity: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
- Existence of a neutral element: $e \cdot a = a$.
- Existence of an inverse: $a^{-1} \cdot a = e$.

Definition of the Lorentz group: Matrix group, which leaves the metric tensor $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ invariant:

$$\Lambda^T g \Lambda = g,$$

The same equation with indices:

$$\Lambda^\mu_\sigma g_{\mu\nu} \Lambda^\nu_\tau = g_{\sigma\tau}.$$

This group is denoted $O(1,3)$. It is easy to show that

$$(\det \Lambda)^2 = 1,$$

and therefore

$$\det \Lambda = \pm 1.$$

If we have in addition $\det \Lambda = 1$ the corresponding group is called the “proper” Lorentz group and denoted $SO(1,3)$.

One further distinguishes the cases whether the time direction is conserved or reversed. If

$$\Lambda^0_0 \geq 1,$$

the time direction is conserved and the corresponding group is called the orthochronous Lorentz group. If on the other hand

$$\Lambda^0_0 \leq -1,$$

then the time direction is reversed. Remark:

$$|\Lambda^0_0| \geq 1$$

follows from $\Lambda^\mu_\sigma g_{\mu\nu} \Lambda^\nu_\tau = g_{\sigma\tau}$ for $\sigma = \tau = 0$:

$$(\Lambda^0_0)^2 - \sum_{j=1}^3 (\Lambda^j_0)^2 = 1.$$

To summarize: The Lorentz group consists of four components, depending on which values the quantities

$$\det \Lambda \quad \text{and} \quad \Lambda^0_0$$

take. The “proper orthochronous Lorentz group” is defined by

$$\Lambda^\mu_\sigma g_{\mu\nu} \Lambda^\nu_\tau = g_{\sigma\tau}, \quad \det \Lambda = 1, \quad \Lambda^0_0 \geq 1,$$

and contains the identity. Elements of the group correspond to rotations in four-dimensional Minkowski space. Each rotation can be decomposed into rotations in the planes xy , yz , zx , tx , ty and tz . A spatial rotation in the xy -plane is given by

$$\Lambda^\mu_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and similar for the yz - and zx -planes. A boost in the tx -plane is given by

$$\Lambda^\mu_\nu = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with

$$\sinh\phi = \frac{\frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}} = \beta\gamma, \quad \cosh\phi = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma,$$

where we used the standard abbreviations

$$\beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Elements of the other three components can be obtained from an element of the proper orthochronous Lorentz group and a discrete transformation of time reversal

$$\Lambda^\mu_\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and / or spatial inversion

$$\Lambda^\mu_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Tensors: $T^{\mu_1\mu_2\dots\mu_r}$ is called a tensor if it transforms under Lorentz transformations as

$$T'^{\mu_1\mu_2\dots\mu_r} = \Lambda^{\mu_1}_{\nu_1}\Lambda^{\mu_2}_{\nu_2}\dots\Lambda^{\mu_r}_{\nu_r}T^{\nu_1\nu_2\dots\nu_r}.$$

The number r is called the rank of the tensor.

Pseudo-tensors: Pseudo-tensors transform under elements of the proper orthochronous Lorentz group as tensors. Under the discrete transformations of time reversal and spatial inversion there is however an additional minus sign. Pseudo-tensors of rank zero are called pseudo-scalars, pseudo-tensors of rank one are called axial vectors.

Examples:

Rank 1: Position vector x^μ , momentum vector p^μ .

Rank 2: Metric tensor $g^{\mu\nu}$.

Rank 4: Total anti-symmetric tensor (Levi-Civita tensor) $\epsilon^{\mu\nu\rho\sigma}$. The total anti-symmetric tensor is defined by

$$\begin{aligned}\epsilon_{0123} &= 1, \\ \epsilon_{\mu\nu\rho\sigma} &= 1 \text{ if } (\mu, \nu, \rho, \sigma) \text{ is an even permutation of } (0, 1, 2, 3), \\ \epsilon_{\mu\nu\rho\sigma} &= -1 \text{ if } (\mu, \nu, \rho, \sigma) \text{ is an odd permutation of } (0, 1, 2, 3), \\ \epsilon_{\mu\nu\rho\sigma} &= 0 \text{ otherwise.}\end{aligned}$$

The total anti-symmetric tensor is a pseudo-tensor, its components are unchanged under time reversal and spatial inversion.

3.3 The Poincaré group

The Poincaré group consists of elements of the Lorentz group and translations. The group elements act on four vectors according to the following transformation law :

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} + a^{\mu}.$$

Λ describes rotations in four dimensional space-time (e.g. ordinary rotations on the spacial components plus boosts) whereas a describes translations.

The group multiplication law is given by

$$\{a_1, \Lambda_1\}\{a_2, \Lambda_2\} = \{a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2\}.$$

The generators of the Poincaré group can be realised as differential operators :

$$\begin{aligned}P_{\mu} &= i\partial_{\mu}, \\ M_{\mu\nu} &= i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}).\end{aligned}$$

The algebra of the Poincaré group is given by

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] &= -i(g_{\mu\rho}M_{\nu\sigma} - g_{\nu\rho}M_{\mu\sigma} + g_{\mu\sigma}M_{\rho\nu} - g_{\nu\sigma}M_{\rho\mu}), \\ [M_{\mu\nu}, P_\sigma] &= i(g_{\nu\sigma}P_\mu - g_{\mu\sigma}P_\nu), \\ [P_\mu, P_\nu] &= 0. \end{aligned}$$

The Poincaré algebra is a Lie algebra, but it is not semi-simple, since it has an Abelian non-trivial ideal (P_μ).

Casimir operators are M^2 and W^2 where

$$M^2 = P_\mu P^\mu, \quad W^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma}.$$

W^μ is called the Lubanski-Pauli vector.

4 Review of classical field theory

Hamilton's principle of the least action can be generalized to systems with infinite many degrees of freedom:

- Countable degrees of freedom q_i ; Example: Model of an elastic rod, described by points of mass m separated by a distance a and connected by massless springs with spring constant k .

$$\begin{aligned} T &= \frac{1}{2}\sum_i m\dot{q}_i^2, \\ V &= \frac{1}{2}\sum_i k(q_{i+1} - q_i)^2. \end{aligned}$$

Lagrange function:

$$L = T - V = \frac{1}{2}\sum_i \left(m\dot{q}_i^2 - k(q_{i+1} - q_i)^2 \right) = \frac{1}{2}\sum_i a \left(\frac{m}{a}\dot{q}_i^2 - ka \left(\frac{q_{i+1} - q_i}{a} \right)^2 \right)$$

- If we have more than countable coordinates q_x we will also use the notation $q(x)$; In the example above we take the limit $a \rightarrow 0$ and we set

$$\begin{aligned} \lim_{a \rightarrow 0} \frac{m}{a} &= \mu, \\ \lim_{a \rightarrow 0} ka &= Y. \end{aligned}$$

Further

$$\lim_{a \rightarrow 0} \frac{q_{i+1} - q_i}{a} = \lim_{a \rightarrow 0} \frac{q(x+a) - q(x)}{a} = \frac{dq(x)}{dx}$$

and we obtain for the Lagrange function

$$L = \frac{1}{2} \int dx \left(\mu \dot{q}^2 - Y \left(\frac{dq(x)}{dx} \right)^2 \right)$$

The expression

$$\mathcal{L} = \frac{1}{2} \left(\mu \dot{q}^2 - Y \left(\frac{dq(x)}{dx} \right)^2 \right)$$

is called the Lagrange density. The action is given by

$$S = \int dt \int dx \mathcal{L}.$$

We now consider the generalization to four-dimensional space-time. We assume that the Lagrange density is a function of the field $\psi(x)$ and its first derivative $\partial_\mu \psi(x)$.

$$\mathcal{L}(\psi(x), \partial_\mu \psi(x))$$

The action is given by

$$S = \int_{t_1}^{t_2} dt \int \int \int d^3x \mathcal{L} = \frac{1}{c} \int d^4x \mathcal{L}$$

From the principle of least action

$$\delta S = 0,$$

we can derive the Euler-Lagrange equations as in classical mechanics, if we assume that the variation of the field vanishes on two hyper-surfaces $t = t_1$ and $t = t_2$:

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} dt \int \int \int d^3x \left[\frac{\partial \mathcal{L}}{\partial \psi} \delta \psi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi(x))} \delta (\partial_\mu \psi(x)) \right] \\ &= \int_{t_1}^{t_2} dt \int \int \int d^3x \left[\frac{\partial \mathcal{L}}{\partial \psi} \delta \psi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi(x))} \partial_\mu (\delta \psi(x)) \right] \\ &= \int_{t_1}^{t_2} dt \int \int \int d^3x \left[\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi(x))} \right] \delta \psi \end{aligned}$$

Therefore

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi(x))} = 0.$$

4.1 The Klein-Gordon field

The Lagrange density for a real scalar field:

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2.$$

The corresponding Euler-Lagrange equation yields the Klein-Gordon equation:

$$(\square + m^2) \phi = 0.$$

The Lagrange density for a complex scalar field reads

$$\mathcal{L}(\phi, \phi^*, \partial_\mu \phi, \partial_\mu \phi^*) = (\partial_\mu \phi^*) (\partial^\mu \phi) - m^2 \phi^* \phi.$$

4.2 The Dirac field

The Lagrange density for the Dirac field depends on four-component spinors $\psi_\alpha(x)$ ($\alpha = 1, 2, 3, 4$) and $\bar{\psi}_\alpha(x) = (\psi^\dagger(x) \gamma^0)_\alpha$:

$$\mathcal{L}(\psi, \bar{\psi}, \partial_\mu \psi) = i \bar{\psi} \gamma^\mu \partial_\mu \psi - m \bar{\psi} \psi$$

Here, the (4×4) -Dirac matrices satisfy the anti-commutation rules

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \mathbf{1}, \quad \{\gamma^\mu, \gamma_5\} = 0, \quad \gamma_5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \frac{i}{24} \varepsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma.$$

Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

4-dimensional σ^μ -matrices:

$$\sigma_{A\dot{B}}^\mu = (1, -\vec{\sigma}) \quad \bar{\sigma}^{\mu\dot{A}B} = (1, \vec{\sigma})$$

Weyl representation for the Dirac matrices:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad \gamma_5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Euler-Lagrange equations yield the Dirac equations

$$\begin{aligned} (i\gamma^\mu \partial_\mu - m) \psi &= 0, \\ \bar{\psi} \left(i\overleftarrow{\gamma}^\mu \partial_\mu + m \right) &= 0. \end{aligned}$$

4.3 The Maxwell field

Lagrange density:

$$\mathcal{L}(A_\mu, \partial_\mu A_\nu) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Then Maxwell's equation follows from the Euler-Lagrange equation

$$\frac{\delta\mathcal{L}}{\delta A_\mu} - \partial_\nu \frac{\delta\mathcal{L}}{\delta(\partial_\nu A_\mu)} = 0.$$

This yields

$$\begin{aligned}\partial_\mu F^{\mu\nu} &= 0, \\ \square A^\mu - \partial^\mu \partial_\nu A^\nu &= 0.\end{aligned}$$

Remark: The first set of Maxwell's equations

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$$

is fulfilled identically with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Gauge invariance: If $A^\mu = (\phi, \vec{A})$, then the electric and magnetic fields are given by

$$\begin{aligned}\vec{B} &= \vec{\nabla} \times \vec{A}, \\ \vec{E} &= -\vec{\nabla}\phi - \frac{\partial}{\partial t}\vec{A}.\end{aligned}$$

Therefore, the potential A_μ determines the electric and magnetic fields. We may ask the reverse question: Given \vec{E} and \vec{B} , does this define uniquely the potential A_μ ? This is not the case. We may add to A_μ the divergence of an arbitrary function:

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \Lambda(x).$$

Since

$$\partial_\mu \partial_\nu \Lambda - \partial_\nu \partial_\mu \Lambda = 0,$$

this leaves the field strength

$$F^{\mu\nu} = \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & -B^z & B^y \\ -E^y & B^z & 0 & -B^x \\ -E^z & -B^y & B^x & 0 \end{pmatrix}$$

and hence the electric and magnetic fields unchanged. Therefore we may impose additional conditions on the gauge potential. A common choice is the covariant Lorenz gauge

$$\partial_\mu A^\mu = 0.$$

A variational problem in the presence of constraints is solved with Lagrange multipliers. This leads us to the Lagrange density

$$\mathcal{L}(A_\mu, \partial_\mu A_\nu) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2$$

and the equation of motion

$$\begin{aligned} \square A^\mu - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial_\nu A^\nu &= 0, \\ \left[\square g_{\mu\nu} - \left(1 - \frac{1}{\xi}\right) \partial_\mu \partial_\nu \right] A^\nu &= 0. \end{aligned}$$

5 Quantum field theory: The canonical formalism

Classical physics: Classical point-like particles and classical fields.

Quantum mechanics: Particles described by a wave function (“first quantization”).

Quantum field theory: Fields described by operators (“second quantization”).

5.1 The Klein-Gordon field as harmonic oscillators

We start with the simplest example, a real scalar field as a classical field with Lagrange density

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2.$$

We define the momentum density conjugate to $\phi(x)$ by

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)}$$

For the Klein-Gordon field we find

$$\pi(x) = \dot{\phi}(x).$$

The Hamiltonian is given by

$$H = \int d^3x \left[\pi(x) \dot{\phi}(x) - \mathcal{L} \right] = \int d^3x \left[\frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 \right] = \int d^3x \mathcal{H}.$$

The energy-momentum tensor is given by

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu} \mathcal{L} = (\partial^\mu \phi) (\partial^\nu \phi) - \frac{1}{2} g^{\mu\nu} (\partial_\rho \phi) (\partial^\rho \phi) + \frac{1}{2} g^{\mu\nu} m^2 \phi^2$$

Note that

$$H = \int d^3x T^{00} = \int d^3x \mathcal{H}$$

and

$$P^i = \int d^3x T^{0i} = - \int d^3x \pi \partial_i \phi.$$

If we expand the classical Klein-Gordon field as

$$\phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \phi(\vec{p}, t),$$

then the Klein-Gordon equation becomes

$$\left[\frac{\partial^2}{\partial t^2} + (|\vec{p}|^2 + m^2) \right] \phi(\vec{p}, t) = 0.$$

This is the same as the equation of a harmonic oscillator with frequency

$$\omega_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2}.$$

For a harmonic oscillator we know how to make the transition from the classical picture to the quantum world. The Hamilton operator for a quantum-mechanical harmonic oscillator with this frequency is given by

$$H = \frac{1}{2}\pi^2 + \frac{1}{2}\omega_{\vec{p}}^2\phi^2.$$

The corresponding creation and annihilation operators are

$$a_{\vec{p}} = \sqrt{\frac{\omega_{\vec{p}}}{2}}\phi + \frac{i}{\sqrt{2\omega_{\vec{p}}}}\pi, \quad a_{\vec{p}}^\dagger = \sqrt{\frac{\omega_{\vec{p}}}{2}}\phi - \frac{i}{\sqrt{2\omega_{\vec{p}}}}\pi.$$

We may solve these equations for ϕ and π and obtain

$$\phi = \frac{1}{\sqrt{2\omega_{\vec{p}}}}(a_{\vec{p}} + a_{\vec{p}}^\dagger), \quad \pi = -i\sqrt{\frac{\omega_{\vec{p}}}{2}}(a_{\vec{p}} - a_{\vec{p}}^\dagger).$$

5.2 The Schrödinger picture

Quantization: Let us start in the Schrödinger picture, where the operators do not depend on time. We start from

$$\begin{aligned} [q_i, p_j] &= i\delta_{ij}, \\ [q_i, q_j] &= [p_i, p_j] = 0, \end{aligned}$$

which for a continuous system becomes

$$\begin{aligned} [\phi(\vec{x}), \pi(\vec{y})] &= i\delta^3(\vec{x} - \vec{y}), \\ [\phi(\vec{x}), \phi(\vec{y})] &= [\pi(\vec{x}), \pi(\vec{y})] = 0. \end{aligned}$$

We write

$$\begin{aligned} \phi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} \right), \\ \pi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\vec{p}}}{2}} \left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} \right) \end{aligned}$$

The inverse formulae read

$$a_{\vec{p}} = \int d^3x \left(\sqrt{\frac{\omega_{\vec{p}}}{2}} \phi(\vec{x}) + \frac{i}{\sqrt{2\omega_{\vec{p}}}} \pi(\vec{x}) \right) e^{-i\vec{p}\cdot\vec{x}},$$

$$a_{\vec{p}}^\dagger = \int d^3x \left(\sqrt{\frac{\omega_{\vec{p}}}{2}} \phi(\vec{x}) - \frac{i}{\sqrt{2\omega_{\vec{p}}}} \pi(\vec{x}) \right) e^{i\vec{p}\cdot\vec{x}}.$$

The commutation relation becomes

$$[a_{\vec{p}}, a_{\vec{q}}^\dagger] = (2\pi)^3 \delta^3(\vec{p} - \vec{q}).$$

In detail:

$$\begin{aligned} [a_{\vec{p}}, a_{\vec{q}}^\dagger] &= \left[\int d^3x \left(\sqrt{\frac{\omega_{\vec{p}}}{2}} \phi(\vec{x}) + \frac{i}{\sqrt{2\omega_{\vec{p}}}} \pi(\vec{x}) \right) e^{-i\vec{p}\cdot\vec{x}}, \int d^3y \left(\sqrt{\frac{\omega_{\vec{q}}}{2}} \phi(\vec{y}) - \frac{i}{\sqrt{2\omega_{\vec{q}}}} \pi(\vec{y}) \right) e^{i\vec{q}\cdot\vec{y}} \right] \\ &= \int d^3x \int d^3y e^{-i\vec{p}\cdot\vec{x}} e^{i\vec{q}\cdot\vec{y}} \left(-\frac{i}{2} \sqrt{\frac{\omega_{\vec{p}}}{\omega_{\vec{q}}}} [\phi(\vec{x}), \pi(\vec{y})] + \frac{i}{2} \sqrt{\frac{\omega_{\vec{q}}}{\omega_{\vec{p}}}} [\pi(\vec{x}), \phi(\vec{y})] \right) \\ &= \int d^3x \int d^3y e^{-i\vec{p}\cdot\vec{x}} e^{i\vec{q}\cdot\vec{y}} \left(\frac{1}{2} \sqrt{\frac{\omega_{\vec{p}}}{\omega_{\vec{q}}}} + \frac{1}{2} \sqrt{\frac{\omega_{\vec{q}}}{\omega_{\vec{p}}}} \right) \delta^3(\vec{x} - \vec{y}) \\ &= \frac{1}{2} \left(\sqrt{\frac{\omega_{\vec{p}}}{\omega_{\vec{q}}}} + \sqrt{\frac{\omega_{\vec{q}}}{\omega_{\vec{p}}}} \right) \int d^3x e^{-i(\vec{p}-\vec{q})\cdot\vec{x}} \\ &= \frac{1}{2} \left(\sqrt{\frac{\omega_{\vec{p}}}{\omega_{\vec{q}}}} + \sqrt{\frac{\omega_{\vec{q}}}{\omega_{\vec{p}}}} \right) (2\pi)^3 \delta^3(\vec{p} - \vec{q}) = (2\pi)^3 \delta^3(\vec{p} - \vec{q}). \end{aligned}$$

The Hamiltonian becomes

$$H = \int d^3x \left[\frac{1}{2} \pi^2 + \frac{1}{2} (\nabla\phi)^2 + \frac{1}{2} m^2 \phi^2 \right] = \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}} \left(a_{\vec{p}}^\dagger a_{\vec{p}} + \frac{1}{2} [a_{\vec{p}}, a_{\vec{p}}^\dagger] \right)$$

The second term is proportional to $\delta(0)$ and gives an infinite constant. Such a term can be expected: A single harmonic oscillator has the ground state energy $\frac{1}{2}\omega$, summing over an infinite number of harmonic oscillators yields an infinite ground state energy. As experiments can only measure energy differences from the ground state, we will ignore this term.

Ground state:

$$a_{\vec{p}} |0\rangle = 0 \text{ for all } a_{\vec{p}}.$$

If we drop the infinite constant above, the ground state has energy $E = 0$. All other states can be obtained by acting on $|0\rangle$ with creation operators. For example, the state

$$a_{\vec{p}}^\dagger a_{\vec{q}}^\dagger |0\rangle$$

has energy $\omega_{\vec{p}} + \omega_{\vec{q}}$. Let us look at the total momentum operator:

$$\vec{P} = - \int d^3x \pi(\vec{x}) \vec{\nabla} \phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \vec{p} a_{\vec{p}}^\dagger a_{\vec{p}}.$$

So the operator $a_{\vec{p}}^\dagger$ creates momentum \vec{p} and energy $\omega_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2}$. From now on we will write

$$E_{\vec{p}} = \omega_{\vec{p}} = +\sqrt{|\vec{p}|^2 + m^2}$$

Note that the energy is always positive.

Since $a_{\vec{p}}^\dagger$ and $a_{\vec{q}}^\dagger$ commute, the state $a_{\vec{p}}^\dagger a_{\vec{q}}^\dagger |0\rangle$ is identical to $a_{\vec{q}}^\dagger a_{\vec{p}}^\dagger |0\rangle$. Furthermore, a single mode \vec{p} can contain arbitrarily many particles, just as the harmonic oscillator can be excited to arbitrarily high levels. Therefore the Klein-Gordon particles obey Bose-Einstein statistics.

Summary: The spectrum of the Hamilton operator

$$H = \int d^3x \left[\frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 \right]$$

is obtained from the ground state $|0\rangle$ by successively applying creation operators $a_{\vec{p}}^\dagger$. All states are reached this way.

Normalization: For the ground state we choose the normalization

$$\langle 0|0\rangle = 1.$$

For one-particle state we choose the normalization

$$\langle \vec{p} | \vec{q} \rangle = 2E_{\vec{p}} (2\pi)^3 \delta^3(\vec{p} - \vec{q}).$$

Therefore

$$|\vec{p}\rangle = \sqrt{2E_{\vec{p}}} a_{\vec{p}}^\dagger |0\rangle.$$

Remark: This normalization is Lorentz invariant. Consider the boost

$$\begin{aligned} E' &= \gamma E - \beta \gamma p_3, \\ p'_3 &= -\beta \gamma E + \gamma p_3. \end{aligned}$$

From

$$\delta(f(x) - f(x_0)) = \frac{1}{|f'(x_0)|} \delta(x - x_0)$$

it follows that

$$\begin{aligned}\delta^3(\vec{p} - \vec{q}) &= \delta^3(\vec{p}' - \vec{q}') \cdot \frac{dp'_3}{dp_3} = \delta^3(\vec{p}' - \vec{q}') \gamma \left(1 - \beta \frac{dE}{dp_3}\right) = \delta^3(\vec{p}' - \vec{q}') \gamma \left(1 - \beta \frac{p_3}{E}\right) \\ &= \delta^3(\vec{p}' - \vec{q}') \frac{E'}{E}.\end{aligned}$$

Remark:

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} = \int \frac{d^4p}{(2\pi)^4} (2\pi) \delta(p^2 - m^2) \theta(p_0)$$

is a Lorentz-invariant 3-momentum integral. Therefore, if $f(p)$ is Lorentz-invariant, so is

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} f(p).$$

5.3 The Heisenberg picture

In the Heisenberg picture, the operators ϕ and π are now time-dependent:

$$\begin{aligned}\phi(x) &= e^{iHt} \phi(\vec{x}) e^{-iHt}, \\ \pi(x) &= e^{iHt} \pi(\vec{x}) e^{-iHt}.\end{aligned}$$

From the Heisenberg equation of motion

$$i \frac{\partial}{\partial t} O = [O, H],$$

we find

$$\begin{aligned}i \frac{\partial}{\partial t} \phi(x) &= i\pi(x), \\ i \frac{\partial}{\partial t} \pi(x) &= i(\vec{\nabla}^2 - m^2) \phi(x).\end{aligned}$$

Combining these two results yields

$$\frac{\partial^2}{\partial t^2} \phi = (\vec{\nabla}^2 - m^2) \phi,$$

which is just the Klein-Gordon equation.

For a better understanding we express $\phi(x)$ and $\pi(x)$ in terms of creation and annihilation operators. From

$$[H, a_{\vec{p}}^\dagger] = E_{\vec{p}} a_{\vec{p}}^\dagger, \quad [H, a_{\vec{p}}] = -E_{\vec{p}} a_{\vec{p}},$$

we have

$$Ha_{\vec{p}}^\dagger = a_{\vec{p}}^\dagger (H + E_{\vec{p}}), \quad Ha_{\vec{p}} = a_{\vec{p}} (H - E_{\vec{p}}),$$

and therefore

$$H^n a_{\vec{p}}^\dagger = a_{\vec{p}}^\dagger (H + E_{\vec{p}})^n, \quad H^n a_{\vec{p}} = a_{\vec{p}} (H - E_{\vec{p}})^n.$$

Therefore

$$e^{iHt} a_{\vec{p}}^\dagger e^{-iHt} = a_{\vec{p}}^\dagger e^{iE_{\vec{p}}t}, \quad e^{iHt} a_{\vec{p}} e^{-iHt} = a_{\vec{p}} e^{iE_{\vec{p}}t}.$$

From

$$\begin{aligned} \phi(\vec{x}) &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} \right), \\ \pi(\vec{x}) &= \int \frac{d^3 p}{(2\pi)^3} (-i) \sqrt{\frac{E_{\vec{p}}}{2}} \left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} \right) \end{aligned}$$

we then obtain

$$\begin{aligned} \phi(x) &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} e^{-ip\cdot x} + a_{\vec{p}}^\dagger e^{ip\cdot x} \right) \Big|_{p^0=E_{\vec{p}}}, \\ \pi(x) &= \frac{\partial}{\partial t} \phi(x). \end{aligned}$$

Remark: $a_{\vec{p}}$ and $a_{\vec{p}}^\dagger$ denote always the time-independent Schrödinger-picture ladder operators.

Remark 2: The above equation makes the duality between particle and wave interpretations of the quantum field explicit: On the one hand, $\phi(x)$ is written as a Hilbert space operator, which creates and destroys the particles that are the quanta of the field excitations. On the other hand, $\phi(x)$ is written as a linear combination of plane-wave solutions of the Klein-Gordon equation.

5.3.1 Causality

Let us consider in the Heisenberg picture the amplitude for a particle to propagate from y to x :

$$D(x-y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle.$$

We express $\phi(x)$ and $\phi(y)$ in terms of creation and annihilation operators. Only the term

$$\langle 0 | a_{\vec{p}} a_{\vec{q}}^\dagger | 0 \rangle = (2\pi)^3 \delta^3(\vec{p} - \vec{q})$$

survives:

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip\cdot(x-y)}$$

Remark: This expression is Lorentz-invariant.

Let us first consider the case where the difference $x - y$ is purely in the time-like direction: $x^0 - y^0 = t, \vec{x} - \vec{y} = 0$:

$$D(x-y) = \frac{4\pi}{(2\pi)^3} \int_0^\infty d|\vec{p}| \frac{|\vec{p}|^2}{2\sqrt{|\vec{p}|^2 + m^2}} e^{-it\sqrt{|\vec{p}|^2 + m^2}} = \frac{1}{4\pi^2} \int_m^\infty dE \sqrt{E^2 - m^2} e^{-iEt}$$

$$\stackrel{t \rightarrow \infty}{\sim} e^{-imt}.$$

Let us now consider the case where $x - y$ is purely spatial: $x^0 - y^0 = 0, \vec{x} - \vec{y} = \vec{r}$:

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{i\vec{p}\cdot\vec{r}} = \frac{2\pi}{(2\pi)^3} \int_0^\infty d|\vec{p}| \int_{-1}^1 d\cos\theta \frac{|\vec{p}|^2}{2E_{\vec{p}}} e^{i|\vec{p}|r\cos\theta}$$

$$= \frac{1}{(2\pi)^2} \int_0^\infty d|\vec{p}| \frac{|\vec{p}|^2}{2E_{\vec{p}}} \frac{e^{i|\vec{p}|r} - e^{-i|\vec{p}|r}}{i|\vec{p}|r} = \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^\infty d|\vec{p}| \frac{|\vec{p}| e^{i|\vec{p}|r}}{\sqrt{|\vec{p}|^2 + m^2}}$$

This integral has branch cuts on the imaginary axis starting at $\pm im$. We can define the contour to go around the upper branch cut. With the substitution $\rho = -i|\vec{p}|$ we obtain

$$\frac{1}{4\pi^2 r} \int_m^\infty d\rho \frac{\rho e^{-\rho r}}{\sqrt{\rho^2 - m^2}} \stackrel{r \rightarrow \infty}{\sim} e^{-mr}.$$

We find that the propagation amplitude for space-like distances is exponentially vanishing, but non-zero. Is this a problem with causality? No, to discuss causality we should not ask whether particles can propagate over space-like distances, but whether a measurement performed at one point can affect a measurement at another point whose separation from the first is space-like. The simplest thing to measure is the field $\phi(x)$, so let's have a look at the commutator $[\phi(x), \phi(y)]$, if this commutator vanishes for space-like distances, one measurement cannot affect another one separated at a space-like distance.

$$[\phi(x), \phi(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{q}}}} \left[\left(a_{\vec{p}} e^{-ip\cdot x} + a_{\vec{p}}^\dagger e^{ip\cdot x} \right), \left(a_{\vec{q}} e^{-iq\cdot y} + a_{\vec{q}}^\dagger e^{iq\cdot y} \right) \right]$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right)$$

$$= D(x-y) - D(y-x).$$

When $(x-y)^2 < 0$, we can perform a Lorentz-transformation on the second term (since each term is separately Lorentz-invariant), taking

$$(x-y) \rightarrow -(x-y).$$

The two terms are therefore equal and cancel in the sum. Therefore causality is preserved.

Remark: If $(x - y)^2 > 0$ there is no continuous Lorentz-transformation, which takes $(x - y) \rightarrow -(x - y)$.

5.3.2 The Klein-Gordon propagator

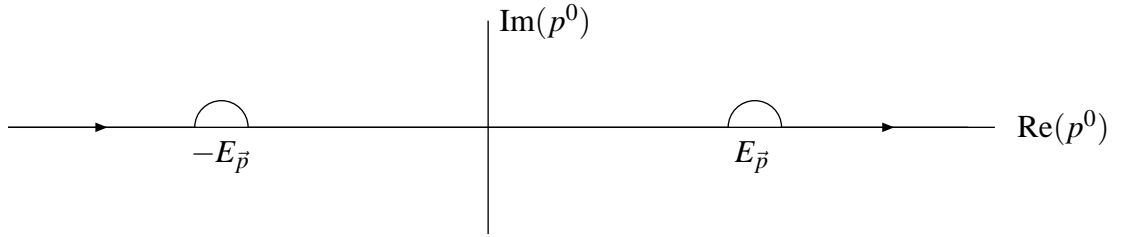
Let us study the commutator $[\phi(x), \phi(y)]$ a little bit further. Since it is a c-number, we have

$$[\phi(x), \phi(y)] = \langle 0 | [\phi(x), \phi(y)] | 0 \rangle.$$

Let us assume that $x^0 > y^0$. Then we have

$$\begin{aligned} \langle 0 | [\phi(x), \phi(y)] | 0 \rangle &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left(e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)} \right) \\ &= \int \frac{d^3 p}{(2\pi)^3} \left\{ \frac{1}{2E_{\vec{p}}} e^{-ip \cdot (x-y)} \Big|_{p^0=E_{\vec{p}}} + \frac{1}{-2E_{\vec{p}}} e^{-ip \cdot (x-y)} \Big|_{p^0=-E_{\vec{p}}} \right\} \\ &= \int \frac{d^3 p}{(2\pi)^3} \int \frac{dp^0}{2\pi i} \frac{-1}{p^2 - m^2} e^{-ip \cdot (x-y)} \end{aligned}$$

In the last step the p^0 integral is to be performed along the contour



To keep track of the contour we also write

$$\langle 0 | [\phi(x), \phi(y)] | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \int \frac{dp^0}{2\pi} \frac{i}{(p^0 + i\varepsilon)^2 - |\vec{p}|^2 - m^2} e^{-ip \cdot (x-y)}$$

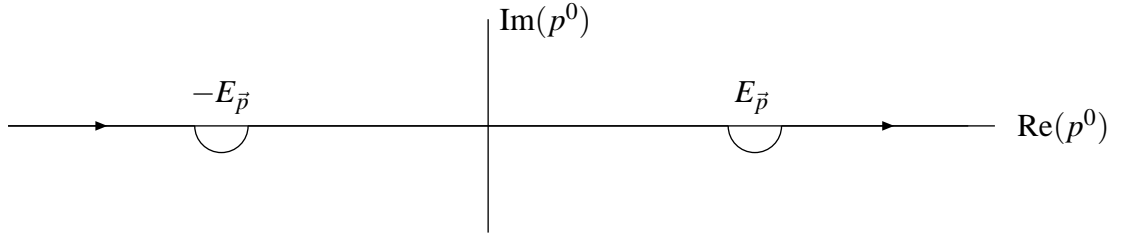
where $\varepsilon > 0$. We define

$$D_R(x-y) = \theta(x^0 - y^0) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{(p^0 + i\varepsilon)^2 - |\vec{p}|^2 - m^2} e^{-ip \cdot (x-y)}$$

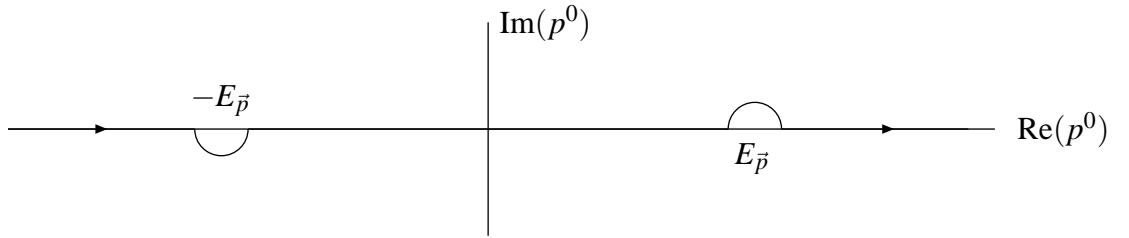
$D_R(x-y)$ is called the retarded Green's function. Similar the advanced Green's function is defined by

$$D_A(x-y) = -\theta(y^0 - x^0) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{(p^0 - i\varepsilon)^2 - |\vec{p}|^2 - m^2} e^{-ip \cdot (x-y)},$$

where in the last expression the contour integral is now evaluated with the contour



There are four possible contours to evaluate the p^0 . A third one is of particular importance:



If $x^0 > y^0$ we can close the contour below and obtain $D(x-y)$. If $x^0 < y^0$ we close the contour above and obtain $D(y-x)$. Therefore

$$\begin{aligned}
 D_F(x-y) &= \begin{cases} D(x-y) & \text{for } x^0 > y^0 \\ D(y-x) & \text{for } x^0 < y^0 \end{cases} \\
 &= \theta(x^0 - y^0) \langle 0 | \phi(x) \phi(y) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | \phi(y) \phi(x) | 0 \rangle \\
 &= \langle 0 | T \phi(x) \phi(y) | 0 \rangle
 \end{aligned}$$

In the last expression the time-ordering symbol T occurs. $D_F(x-y)$ is called the Feynman propagator. We have

$$D_F(x-y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)},$$

Let's have a look at

$$\begin{aligned}
 (\square + m^2) D_F(x-y) &= (\square + m^2) \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip \cdot (x-y)} \\
 &= \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2} (-p^2 + m^2) e^{-ip \cdot (x-y)} \\
 &= -i\delta^4(x-y).
 \end{aligned}$$

Therefore, if we look at the Fourier transform $\tilde{D}_F(p)$

$$D_F(x-y) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x-y)} \tilde{D}_F(p),$$

we obtain an algebraic equation for $\tilde{D}_F(p)$:

$$(p^2 - m^2)\tilde{D}_F(p) = i.$$

Summary: The Feynman propagator in momentum space is obtained from an algebraic equation. The integration contour in the p^0 -plane is given by the $i\varepsilon$ -prescription. For the Klein-Gordon propagator we have

$$\tilde{D}_F(p) = \frac{i}{p^2 - m^2 + i\varepsilon}.$$

5.4 Wick's theorem

We already had the definition of the time-ordered product: This product orders operators such that the time does not decrease from right to left.

$$T\phi(x)\phi(y) = \begin{cases} \phi(x)\phi(y) & \text{for } x^0 > y^0 \\ \phi(y)\phi(x) & \text{for } y^0 > x^0 \end{cases}$$

In addition we introduce the normal product, which orders operators such that all annihilation operators are on the right of all creation operators:

$$\begin{aligned} :a_p^\dagger a_{\bar{q}}: &= a_p^\dagger a_{\bar{q}}, \\ :a_{\bar{q}} a_p^\dagger: &= a_p^\dagger a_{\bar{q}}. \end{aligned}$$

We notice that the vacuum expectation value of a normal ordered product is zero (unless the product is empty):

$$\langle 0 | : \phi(x_1)\phi(x_2)\dots\phi(x_n) : | 0 \rangle = 0.$$

We further introduce the so-called ‘‘contraction’’, which is just the vacuum expectation value of the time-ordered product of two operators:

$$\underline{\phi(x)\phi(y)} = \langle 0 | T\phi(x)\phi(y) | 0 \rangle.$$

Wick's theorem states that

$$T\phi(x_1)\phi(x_2)\dots\phi(x_n) = : \phi(x_1)\phi(x_2)\dots\phi(x_n) : + \text{all possible contractions.}$$

Example:

$$\begin{aligned} T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) &= : \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) : \\ &+ : \underline{\phi(x_1)\phi(x_2)}\phi(x_3)\phi(x_4) : + : \underline{\phi(x_1)\phi(x_2)}\phi(x_3)\phi(x_4) : + : \underline{\phi(x_1)\phi(x_2)}\phi(x_3)\phi(x_4) : \\ &+ : \phi(x_1)\phi(x_2)\underline{\phi(x_3)\phi(x_4)} : + : \phi(x_1)\phi(x_2)\underline{\phi(x_3)\phi(x_4)} : + : \phi(x_1)\phi(x_2)\underline{\phi(x_3)\phi(x_4)} : \\ &+ : \phi(x_1)\phi(x_2)\phi(x_3)\underline{\phi(x_4)} : + : \phi(x_1)\phi(x_2)\phi(x_3)\underline{\phi(x_4)} : + : \phi(x_1)\phi(x_2)\phi(x_3)\underline{\phi(x_4)} : \end{aligned}$$

Proof: We decompose any operator into positive and negative frequency parts:

$$\begin{aligned}\phi(x) &= \phi^+(x) + \phi^-(x), \\ \phi^+(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} a_{\vec{p}} e^{-ip \cdot x}, \quad \phi^-(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} a_{\vec{p}}^\dagger e^{ip \cdot x}.\end{aligned}$$

ϕ^+ contains only annihilation operators, ϕ^- contains only creation operators. We proof Wick's theorem by induction. We start at $n = 2$ and assume $x^0 > y^0$:

$$\begin{aligned}T\phi(x)\phi(y) &= \phi(x)\phi(y) \\ &= \phi^+(x)\phi^+(y) + \phi^+(x)\phi^-(y) + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y) \\ &= \phi^+(x)\phi^+(y) + \phi^-(y)\phi^+(x) + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y) + [\phi^+(x), \phi^-(y)] \\ &= : \phi(x)\phi(y) : + [\phi^+(x), \phi^-(y)] \\ &= : \phi(x)\phi(y) : + \langle 0 | [\phi^+(x), \phi^-(y)] | 0 \rangle \\ &= : \phi(x)\phi(y) : + D_F(x-y) \\ &= : \phi(x)\phi(y) : + \underline{\phi(x)\phi(y)}.\end{aligned}$$

Not let's assume that it is valid for $n - 1$ fields. Again we assume $x_1^0 > x_2^0 > \dots > x_n^0$.

$$\begin{aligned}T\phi(x_1)\phi(x_2)\dots\phi(x_n) &= \phi(x_1)\phi(x_2)\dots\phi(x_n) \\ &= \phi(x_1) : \{ \phi(x_2)\dots\phi(x_n) + \text{all contractions not involving } \phi(x_1) \} : \\ &= (\phi^+(x_1) + \phi^-(x_1)) : \{ \phi(x_2)\dots\phi(x_n) + \text{all contractions not involving } \phi(x_1) \} : \end{aligned}$$

We want to move $\phi^+(x_1)$ and $\phi^-(x_1)$ inside the normal product. For $\phi^-(x_1)$ this is easy: $\phi^-(x_1)$ contains only creation operators, therefore

$$\phi^-(x_1) : \phi(x_2)\dots\phi(x_n) : = : \phi^-(x_1)\phi(x_2)\dots\phi(x_n) :,$$

and we can just move it in to the left of all other operators. On the other hand we have

$$: \phi^+(x_1)\phi(x_2)\dots\phi(x_n) : = : \phi(x_2)\dots\phi(x_n)\phi^+(x_1) :,$$

therefore

$$\begin{aligned}\phi^+(x_1) : \phi(x_2)\dots\phi(x_n) : &=: \phi(x_2)\phi(x_3)\dots\phi(x_n) : \phi^+(x_1) + [\phi^+(x_1), : \phi(x_2)\phi(x_3)\dots\phi(x_n) :] \\ &= : \phi^+(x_1)\phi(x_2)\phi(x_3)\dots\phi(x_n) : + : [\phi^+(x_1), \phi(x_2)] \phi(x_3)\dots\phi(x_n) : + : \phi(x_2) [\phi^+(x_1), \phi(x_3)] \dots\phi(x_n) : \\ &\quad + : \phi(x_2)\phi(x_3)\dots [\phi^+(x_1), \phi(x_n)] : \\ &= : \phi^+(x_1)\phi(x_2)\phi(x_3)\dots\phi(x_n) : + : \underline{\phi^+(x_1)\phi(x_2)\phi(x_3)\dots\phi(x_n)} : + : \underline{\phi^+(x_1)\phi(x_2)\phi(x_3)\dots\phi(x_n)} : + \dots,\end{aligned}$$

which proves Wick's theorem. Let us now apply Wick's theorem to the vacuum expectation value of

$$\langle 0 | T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) | 0 \rangle.$$

By construction, the vacuum expectation value of a non-empty normal product vanishes, therefore

$$\begin{aligned}
\langle 0 | T \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) | 0 \rangle &= \\
&= \left\langle 0 \left| : \underbrace{\phi(x_1) \phi(x_2)} \underbrace{\phi(x_3) \phi(x_4)} : \right| 0 \right\rangle + \left\langle 0 \left| : \underbrace{\phi(x_1) \phi(x_2) \phi(x_3)} \phi(x_4) : \right| 0 \right\rangle \\
&\quad + \left\langle 0 \left| : \underbrace{\phi(x_1) \phi(x_2) \phi(x_4)} \phi(x_3) : \right| 0 \right\rangle \\
&= \langle 0 | T \phi(x_1) \phi(x_2) | 0 \rangle \langle 0 | T \phi(x_3) \phi(x_4) | 0 \rangle + \langle 0 | T \phi(x_1) \phi(x_3) | 0 \rangle \langle 0 | T \phi(x_2) \phi(x_4) | 0 \rangle \\
&\quad + \langle 0 | T \phi(x_1) \phi(x_4) | 0 \rangle \langle 0 | T \phi(x_2) \phi(x_3) | 0 \rangle \\
&= D_F(x_1 - x_2) D_F(x_3 - x_4) + D_F(x_1 - x_3) D_F(x_2 - x_4) + D_F(x_1 - x_4) D_F(x_2 - x_3).
\end{aligned}$$

Graphically,

$$\langle 0 | T \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) | 0 \rangle = \begin{array}{c} 1 \text{ --- } 2 \\ \bullet \text{ --- } \bullet \\ 4 \text{ --- } 3 \\ \bullet \text{ --- } \bullet \end{array} + \begin{array}{c} 1 \\ | \\ \bullet \\ | \\ \bullet \\ | \\ 4 \end{array} \begin{array}{c} 2 \\ | \\ \bullet \\ | \\ \bullet \\ | \\ 3 \end{array} + \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ 4 \quad 3 \end{array}$$

5.5 Interacting fields

Up to now we considered “free” fields, e.g. fields without any interactions. For the free Klein-Gordon field we had the Lagrange density

$$\mathcal{L}_0 = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2.$$

In this theory, no interactions and no scattering occurs. Let us start to look at more interesting theories with interactions:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4,$$

where λ is a dimensionless coupling constant. This theory is often called “phi-fourth” theory and one of the simplest theories with interactions. Obviously,

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}, \quad \mathcal{L}_{int} = -\frac{\lambda}{4!} \phi^4.$$

The “classical” equation of motion for the ϕ^4 theory is

$$(\square + m^2) \phi = -\frac{\lambda}{3!} \phi^3,$$

which cannot be solved by Fourier analysis as the free Klein-Gordon equation. For $\lambda = 0$ we recover the Klein-Gordon equation. If λ is small we may treat the interacting theory by perturbation theory.

As \mathcal{L}_{int} does not involve any derivatives, the definition of

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

is unaffected by \mathcal{L}_{int} . With the same reasoning, we still impose in the quantum theory equal-time commutation relations

$$[\phi(\vec{x}), \pi(\vec{y})] = i\delta^3(\vec{x} - \vec{y}).$$

We can write the full Hamiltonian as

$$\begin{aligned} H &= H_0 + H_{int}, \\ H_0 &= \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right], \\ H_{int} &= \int d^3x \frac{\lambda}{4!}\phi^4. \end{aligned}$$

We start with the study of the two-point correlation function, or the two-point Green's function

$$\langle \Omega | T\phi(x)\phi(y) | \Omega \rangle$$

$|\Omega\rangle$ denotes the ground state of the interacting theory, which is in general different from the ground state $|0\rangle$ of the free theory. The interaction Hamiltonian enters in two places: In the definition of $|\Omega\rangle$ and in the definition of the Heisenberg field

$$\phi(x) = e^{iHt}\phi(\vec{x})e^{-iHt}.$$

It is easiest to begin with the Heisenberg field $\phi(x)$. At any fixed time t_0 we can of course expand the field as before in terms of creation and annihilation operators:

$$\phi(t_0, \vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} \right).$$

For $t \neq t_0$ we have in the Heisenberg picture

$$\phi(t, \vec{x}) = e^{iH(t-t_0)}\phi(t_0, \vec{x})e^{-iH(t-t_0)}.$$

For $\lambda = 0$, H reduces to H_0 :

$$\phi(t, \vec{x})|_{\lambda=0} = e^{iH_0(t-t_0)}\phi(t_0, \vec{x})e^{-iH_0(t-t_0)}.$$

When λ is small, this expression will still give the most important part of the time dependence of $\phi(x)$, and thus it is convenient to give this quantity a name: the interaction picture field, $\phi_I(x)$. As in the free theory we find

$$\phi_I(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} e^{-ip\cdot x} + a_{\vec{p}}^\dagger e^{ip\cdot x} \right) \Big|_{x^0=t-t_0}.$$

The problem is now to express the full Heisenberg field $\phi(x)$ in terms of $\phi_I(x)$. We have

$$\begin{aligned}\phi(x) &= e^{iH(t-t_0)} e^{-iH_0(t-t_0)} \phi_I(x) e^{iH_0(t-t_0)} e^{-iH(t-t_0)} \\ &= U^\dagger(t, t_0) \phi_I(x) U(t, t_0),\end{aligned}$$

where

$$U(t, t_0) = e^{iH_0(t-t_0)} e^{-iH(t-t_0)}.$$

We have

$$U(t_0, t_0) = 1$$

and

$$\begin{aligned}i \frac{\partial}{\partial t} U(t, t_0) &= e^{iH_0(t-t_0)} (H - H_0) e^{-iH(t-t_0)} \\ &= e^{iH_0(t-t_0)} H_{int} e^{-iH(t-t_0)} \\ &= \underbrace{e^{iH_0(t-t_0)} H_{int} e^{-iH_0(t-t_0)}}_{H_I(t)} \underbrace{e^{iH_0(t-t_0)} e^{-iH(t-t_0)}}_{U(t, t_0)}\end{aligned}$$

$H_I(t)$ is the interaction Hamiltonian written in the interaction picture:

$$H_I(t) = e^{iH_0(t-t_0)} H_{int} e^{-iH_0(t-t_0)} = \int d^3x \frac{\lambda}{4!} \phi_I^4.$$

Therefore

$$i \frac{\partial}{\partial t} U(t, t_0) = H_I(t) U(t, t_0), \quad U(t_0, t_0) = 1.$$

A solution is given by

$$\begin{aligned}U(t, t_0) &= 1 + (-i) \int_{t_0}^t dt_1 H_I(t_1) + (-i)^2 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1) H_I(t_2) \\ &\quad + (-i)^3 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 H_I(t_1) H_I(t_2) H_I(t_3) + \dots\end{aligned}$$

This solution can be verified by differentiation. The initial condition $U(t_0, t_0) = 1$ is obviously satisfied. Note that the various factors of H_I stand in time order, later on the left. Note that

$$\int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1) H_I(t_2) = \frac{1}{2} \int_{t_0}^t dt_1 \int_{t_0}^t dt_2 T(H_I(t_1) H_I(t_2))$$

Similar

$$\int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n H_I(t_1) H_I(t_2) \dots H_I(t_n) = \frac{1}{n!} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n T(H_I(t_1) H_I(t_2) \dots H_I(t_n)).$$

Therefore

$$\begin{aligned} U(t, t_0) &= 1 + (-i) \int_{t_0}^t dt_1 H_I(t_1) + \frac{(-i)^2}{2!} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 T(H_I(t_1) H_I(t_2)) \\ &\quad + \frac{(-i)^3}{3!} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 T(H_I(t_1) H_I(t_2) H_I(t_3)) + \dots \\ &= T \left\{ \exp \left[-i \int_{t_0}^t dt' H_I(t') \right] \right\}. \end{aligned}$$

Therefore we have expressed the full field in terms of ϕ_I ;

$$\phi(x) = U^\dagger(t, t_0) \phi_I(x) U(t, t_0),$$

We can generalize the evolution operator to take as the second argument values other than our reference time t_0 :

$$U(t, t') = T \left\{ \exp \left[-i \int_{t'}^t dt'' H_I(t'') \right] \right\}.$$

$U(t, t')$ satisfies

$$i \frac{\partial}{\partial t} U(t, t') = H_I(t) U(t, t'), \quad U(t, t) = 1,$$

and the identities

$$\begin{aligned} U(t_1, t_2) U(t_2, t_3) &= U(t_1, t_3), \\ U(t_1, t_3) U^\dagger(t_2, t_3) &= U(t_1, t_2). \end{aligned}$$

Therefore

$$\begin{aligned} U(t, t') &= U(t, t_0) U^\dagger(t', t_0) \\ &= e^{iH_0(t-t_0)} e^{-iH(t-t')} e^{-iH_0(t'-t_0)}, \end{aligned}$$

which proves that $U(t, t')$ is unitary. In particular we have

$$U(t_0, -T) = e^{-iH(t_0-(-T))} e^{-iH_0((-T)-t_0)}.$$

Let us now discuss the ground state $|\Omega\rangle$. Imagine starting with $|0\rangle$, the ground state of H_0 and evolving through time with H :

$$e^{-iHT}|0\rangle = \sum_n e^{-i\hat{E}_n T} |\hat{n}\rangle \langle \hat{n}|0\rangle$$

Here we inserted a complete set of states. \hat{E}_n and $|\hat{n}\rangle$ are the eigenvalues and eigenstates of the full Hamiltonian H . We define the zero of the energy by

$$H_0|0\rangle = 0.$$

We single out the ground state $|\Omega\rangle = |\hat{0}\rangle$:

$$e^{-iHT}|0\rangle = e^{-i\hat{E}_0 T} |\Omega\rangle \langle \Omega|0\rangle + \sum_{n \neq 0} e^{-i\hat{E}_n T} |\hat{n}\rangle \langle \hat{n}|0\rangle,$$

where $\hat{E}_0 = \langle \Omega|H|\Omega\rangle$. Since $\hat{E}_n > \hat{E}_0$ for $n > 0$ the additional terms disappear, if we send $T \rightarrow \infty(1-i\epsilon)$. Therefore

$$|\Omega\rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \left(e^{-i\hat{E}_0 T} \langle \Omega|0\rangle \right)^{-1} e^{-iHT}|0\rangle$$

Since T is now very large, we can shift it by a small constant:

$$\begin{aligned} |\Omega\rangle &= \lim_{T \rightarrow \infty(1-i\epsilon)} \left(e^{-i\hat{E}_0(T+t_0)} \langle \Omega|0\rangle \right)^{-1} e^{-iH(T+t_0)}|0\rangle \\ &= \lim_{T \rightarrow \infty(1-i\epsilon)} \left(e^{-i\hat{E}_0(t_0-(-T))} \langle \Omega|0\rangle \right)^{-1} e^{-iH(t_0-(-T))} e^{-iH_0((-T)-t_0)}|0\rangle \\ &= \lim_{T \rightarrow \infty(1-i\epsilon)} \left(e^{-i\hat{E}_0(t_0-(-T))} \langle \Omega|0\rangle \right)^{-1} U(t_0, -T)|0\rangle \end{aligned}$$

In the second line we have used $H_0|0\rangle = 0$. Similar, we can express $\langle \Omega|$ as

$$\langle \Omega| = \lim_{T \rightarrow \infty(1-i\epsilon)} \langle 0|U(T, t_0) \left(e^{-i\hat{E}_0(T-t_0)} \langle 0|\Omega\rangle \right)^{-1}.$$

Therefore we have for $x^0 > y^0 > t_0$:

$$\begin{aligned} \langle \Omega|T\phi(x)\phi(y)|\Omega\rangle &= \\ &= \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\langle 0|U(T, t_0)U^\dagger(x^0, t_0)\phi_I(x)U(x^0, t_0)U^\dagger(y^0, t_0)\phi_I(x)U(y^0, t_0)U(t_0, -T)|0\rangle}{e^{-2i\hat{E}_0 T} \langle \Omega|0\rangle \langle 0|\Omega\rangle} \\ &= \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\langle 0|U(T, x^0)\phi_I(x)U(x^0, y^0)\phi_I(x)U(y^0, -T)|0\rangle}{e^{-2i\hat{E}_0 T} |\langle 0|\Omega\rangle|^2} \end{aligned}$$

Now

$$1 = \langle \Omega|\Omega\rangle = \frac{\langle 0|U(T, t_0)U(t_0, -T)|0\rangle}{e^{-2i\hat{E}_0 T} |\langle 0|\Omega\rangle|^2} = \frac{\langle 0|U(T, -T)|0\rangle}{e^{-2i\hat{E}_0 T} |\langle 0|\Omega\rangle|^2}$$

Therefore we have for $x^0 > y^0$

$$\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle = \lim_{T \rightarrow \infty (1-i\epsilon)} \frac{\langle 0 | U(T, x^0) \phi_I(x) U(x^0, y^0) \phi_I(y) U(y^0, -T) | 0 \rangle}{\langle 0 | U(T, -T) | 0 \rangle}$$

We notice that all fields are in time-order. The final formula, valid for any x^0 and y^0 reads

$$\begin{aligned} \langle \Omega | T \phi(x) \phi(y) | \Omega \rangle &= \lim_{T \rightarrow \infty (1-i\epsilon)} \frac{\langle 0 | T \{ \phi_I(x) \phi_I(y) U(T, -T) \} | 0 \rangle}{\langle 0 | U(T, -T) | 0 \rangle} \\ &= \lim_{T \rightarrow \infty (1-i\epsilon)} \frac{\left\langle 0 \left| T \left\{ \phi_I(x) \phi_I(y) \exp \left[-i \int_{-T}^T dt H_I(t) \right] \right\} \right| 0 \right\rangle}{\left\langle 0 \left| T \left\{ \exp \left[-i \int_{-T}^T dt H_I(t) \right] \right\} \right| 0 \right\rangle} \end{aligned}$$

5.6 Feynman diagrams

We have already seen in the free field theory, that we can represent certain expressions graphically, e.g.

$$\langle 0 | T \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) | 0 \rangle = \begin{array}{c} \begin{array}{ccc} 1 & \text{---} & 2 \\ \bullet & & \bullet \\ 4 & \text{---} & 3 \\ \bullet & & \bullet \end{array} & + & \begin{array}{ccc} 1 & & 2 \\ \bullet & & \bullet \\ 4 & & 3 \\ \bullet & & \bullet \end{array} & + & \begin{array}{ccc} 1 & & 2 \\ \bullet & & \bullet \\ 4 & & 3 \\ \bullet & & \bullet \end{array} & + & \begin{array}{ccc} 1 & & 2 \\ \bullet & & \bullet \\ 4 & & 3 \\ \bullet & & \bullet \end{array} \end{array}$$

This describes two non-interacting particles. Things get more interesting in the interacting theory as soon as we have more than one field at the same space-time point. Let's again look at

$$\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle = \lim_{T \rightarrow \infty (1-i\epsilon)} \frac{\left\langle 0 \left| T \left\{ \phi_I(x) \phi_I(y) \exp \left[-i \int_{-T}^T dt H_I(t) \right] \right\} \right| 0 \right\rangle}{\left\langle 0 \left| T \left\{ \exp \left[-i \int_{-T}^T dt H_I(t) \right] \right\} \right| 0 \right\rangle}$$

in the interacting ϕ^4 -theory. Expanding the exponential in the numerator we obtain

$$\begin{aligned} \left\langle 0 \left| T \left\{ \phi_I(x) \phi_I(y) \exp \left[-i \int_{-T}^T dt H_I(t) \right] \right\} \right| 0 \right\rangle &= \\ & \langle 0 | T \{ \phi_I(x) \phi_I(y) \} | 0 \rangle + \left\langle 0 \left| T \left\{ \phi_I(x) \phi_I(y) \left[-i \int_{-T}^T dt H_I(t) \right] \right\} \right| 0 \right\rangle + \dots \\ &= D_F(x-y) + \left\langle 0 \left| T \left\{ \phi_I(x) \phi_I(y) (-i) \int dt \int d^3 z \frac{\lambda}{4!} \phi_I^4 \right\} \right| 0 \right\rangle + \dots \\ &= D_F(x-y) + \left(\frac{-i\lambda}{4!} \right) \int d^4 z \langle 0 | T \{ \phi_I(x) \phi_I(y) \phi_I(z) \phi_I(z) \phi_I(z) \phi_I(z) \} | 0 \rangle + \dots \end{aligned}$$

For the term proportional to λ we use Wick's theorem. In total there are 15 possibilities of contracting the 6 fields with each other. However, only two of them are really different. The first possibility is to contract $\phi_I(x)$ with $\phi_I(y)$, then we have three possibilities of contracting the four $\phi_I(z)$ with each other, all of them are equivalent. The second possibility consists in contracting $\phi_I(x)$ with one of the $\phi_I(z)$ (four choices) and $\phi_I(y)$ with one of the remaining $\phi_I(z)$ (three choices). The two remaining $\phi_I(z)$ are then contracted with each other. Therefore

$$\langle 0 | T \{ \phi_I(x) \phi_I(y) \phi_I(z) \phi_I(z) \phi_I(z) \phi_I(z) \} | 0 \rangle = 3 D_F(x-y) D_F(z-z) D_F(z-z) + 12 D_F(x-z) D_F(y-z) D_F(z-z).$$

Graphically:

$$3 \left(\begin{array}{c} \bullet \text{---} \bullet \\ x \qquad y \\ \circ \text{---} \circ \\ \qquad z \end{array} \right) + 12 \left(\begin{array}{c} \bullet \text{---} \circ \text{---} \bullet \\ x \qquad z \qquad y \end{array} \right)$$

Lines in this diagram are called propagators, internal points, where four lines meet are called vertices. Therefore

$$\left(\frac{-i\lambda}{4!} \right) \int d^4z \langle 0 | T \{ \phi_I(x) \phi_I(y) \phi_I(z) \phi_I(z) \phi_I(z) \phi_I(z) \} | 0 \rangle = (-i\lambda) \int d^4z \left[\frac{1}{8} D_F(x-y) D_F(z-z) D_F(z-z) + \frac{1}{2} D_F(x-z) D_F(y-z) D_F(z-z) \right].$$

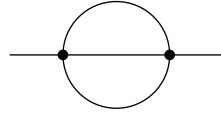
The numbers 8 and 2 are called the symmetry factors of the diagrams. The symmetry factor is given by a factor $4!$ for each vertex in the diagram. One then divides by the number of ways one obtains from Wick's theorem the same diagram (3 and 12 in the above example).

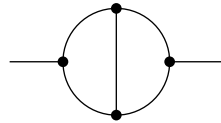
Alternatively, one can compute the symmetry factor as follows: S is the order of the permutation group of the internal lines and vertices leaving the diagram unchanged when the external lines are fixed.

Examples:

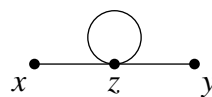
$$\begin{array}{c} \circ \\ \bullet \end{array} \quad S = 2$$

$$\begin{array}{c} \circ \text{---} \circ \\ \bullet \end{array} \quad S = 2 \cdot 2 \cdot 2 = 8$$


 $S = 3! = 6$

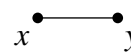

 $S = 2$

It is common practice to let a Feynman diagram represent all possible contractions leading to this diagram with the appropriate symmetry factor included. Therefore



 $= (-i\lambda) \frac{1}{2} \int d^4z D_F(x-z) D_F(y-z) D_F(z-z)$

We can now formulate the Feynman rules for ϕ^4 -theory in position space:

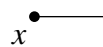
1. For each propagator,


 $= D_F(x-y)$

2. For each vertex,


 $= (-i\lambda) \int d^4z$

3. For each external point,


 $= 1$

4. Divide by the symmetry factor S .

In practice one works in momentum space. As an example we consider the following Fourier transform

$$\int d^4x e^{ipx} \int d^4y e^{iqy} (-i\lambda) \frac{1}{2} \int d^4z D_F(x-z) D_F(y-z) D_F(z-z) =$$

$$\begin{aligned}
& (-i\lambda) \frac{1}{2} \int d^4x \int d^4y \int d^4z e^{i(px+qy)} \int \frac{d^4p_1}{(2\pi)^4} e^{-ip_1(x-z)} \int \frac{d^4p_2}{(2\pi)^4} e^{-ip_2(y-z)} \int \frac{d^4p_3}{(2\pi)^4} e^{-ip_3(z-z)} \\
& \times \tilde{D}_F(p_1) \tilde{D}_F(p_2) \tilde{D}_F(p_3) \\
= & (-i\lambda) \frac{1}{2} \int d^4p_1 \int d^4p_2 \int d^4p_3 \delta^4(p_1 - p) \delta^4(p_2 - q) \delta^4(p_1 + p_2 - p_3 + p_3) \tilde{D}_F(p_1) \tilde{D}_F(p_2) \tilde{D}_F(p_3) \\
= & (-i\lambda) \frac{1}{2} (2\pi)^4 \delta^4(p + q) \tilde{D}_F(p) \tilde{D}_F(q) \int \frac{d^4p_3}{(2\pi)^4} \tilde{D}_F(p_3)
\end{aligned}$$

We see that we obtain one delta-function, enforcing energy-momentum conservation of the incoming and outgoing momenta. Further at each vertex, energy and momentum are conserved. Finally we have to integrate over any unrestricted four-momentum. We use the notation

$$G^n(x_1, \dots, x_n) = \langle 0 | T(\phi(x_1) \dots \phi(x_n)) | 0 \rangle$$

for the Green function with n external legs and define it's Fourier transform by

$$G^n(x_1, \dots, x_n) = \int \frac{d^4p_1}{(2\pi)^4} \dots \frac{d^4p_n}{(2\pi)^4} e^{-i\sum p_j x_j} (2\pi)^4 \delta(p_1 + \dots + p_n) \tilde{G}^n(p_1, \dots, p_n)$$

Remark: This is not the standard Fourier transform, as we have factored out the term

$$(2\pi)^4 \delta(p_1 + \dots + p_n).$$

This term expresses overall momentum conservation and is always there, therefore it is convenient to factor it out. With these conventions we can now state the Feynman rules in momentum space to compute $\tilde{G}^n(p_1, \dots, p_n)$:

1. For each propagator,

$$\bullet \xrightarrow{p} \bullet = \frac{i}{p^2 - m^2 + i\epsilon}$$

2. For each vertex,

$$\begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array} = (-i\lambda)$$

3. For each external point,

$$\bullet \xleftarrow{p} = 1$$

4. Impose momentum conservation at each vertex.

5. Integrate over each undetermined momentum;

$$\int \frac{d^4 p}{(2\pi)^4}$$

6. Divide by the symmetry factor S .

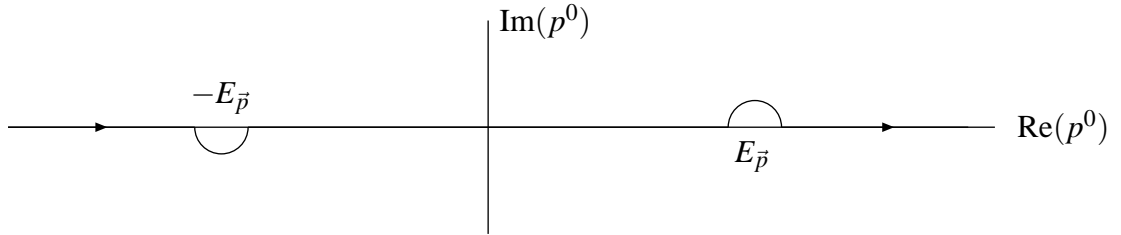
Remark: In the original formula in position space we had

$$\lim_{T \rightarrow \infty(1-i\epsilon)} \int_{-T}^T dt \int d^3 z$$

which we wrote for simplicity as

$$\int d^4 z.$$

Still we have to keep in mind that for the time-component we have to integrate over a slightly distorted contour. When going to momentum space this is equivalent to integrate over the p^0 -component along the following contour:



This is just the Feynman prescription.

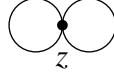
Let us go back to our original formula

$$\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\langle 0 | T \left\{ \phi_I(x) \phi_I(y) \exp \left[-i \int_{-T}^T dt H_I(t) \right] \right\} | 0 \rangle}{\langle 0 | T \left\{ \exp \left[-i \int_{-T}^T dt H_I(t) \right] \right\} | 0 \rangle}$$

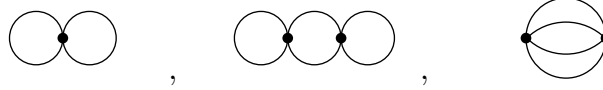
In the numerator we had contributions like

$$(-i\lambda) \int d^4 z \frac{1}{8} D_F(x-y) D_F(z-z) D_F(z-z) = \left(\begin{array}{c} \bullet \text{---} \bullet \\ x \qquad y \end{array} \quad \begin{array}{c} \bigcirc \\ \bullet \\ \bigcirc \\ z \end{array} \right).$$

The piece



is called a vacuum diagram. It is not connected to the external points x and y . Examples for vacuum diagrams are:



It is clear that the denominator in the formula for $\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle$ produces only vacuum graphs. Let us label the values of the various vacuum graphs by V_i , and the values of the pieces connected to x and y by D_j . If a diagram consists of a piece D_j and n_1 vacuum graphs of type V_1 , n_2 vacuum graphs V_2 , ..., then the value for the total diagram is

$$D_j \prod_i \frac{1}{n_i!} (V_i)^{n_i}$$

The factor $n_i!$ is the symmetry factor for n_i copies of V_i . Then the numerator of the formula for the two-point correlation function is

$$\sum_j \sum_{\{n_i\}} D_j \prod_i \frac{1}{n_i!} (V_i)^{n_i},$$

where $\{n_i\}$ means all ordered sets $\{n_1, n_2, \dots\}$ of non-negative integers. We have the following exponentiation:

$$\begin{aligned} \sum_j \sum_{\{n_i\}} D_j \prod_i \frac{1}{n_i!} (V_i)^{n_i} &= \sum_j D_j \prod_i \sum_{n_i} \frac{1}{n_i!} (V_i)^{n_i} \\ &= \sum_j D_j \prod_i \exp(V_i) \\ &= \left(\sum_j D_j \right) \exp \left(\sum_i V_i \right) \end{aligned}$$

On the other hand we find for the denominator

$$\left\langle 0 \left| T \left\{ \exp \left[-i \int_{-T}^T dt H_I(t) \right] \right\} \right| 0 \right\rangle = \exp \left(\sum_i V_i \right).$$

Therefore the exponential cancels between numerator and denominator and we finally obtain

$$\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle = \text{sum of all diagrams without any vacuum graphs.}$$

Summary of the last lecture

ϕ^4 -theory:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4,$$

We are interested in

$$G^n(x_1, \dots, x_n) = \langle \Omega | T(\phi(x_1) \dots \phi(x_n)) | \Omega \rangle$$

and it's Fourier transform defined through

$$G^n(x_1, \dots, x_n) = \int \frac{d^4 p_1}{(2\pi)^4} \dots \frac{d^4 p_n}{(2\pi)^4} e^{-i \sum p_j x_j} (2\pi)^4 \delta(p_1 + \dots + p_n) \tilde{G}^n(p_1, \dots, p_n)$$

Feynman rules in momentum space: $\tilde{G}^n(p_1, \dots, p_n)$ is given as the sum of all diagrams without any vacuum graphs with the following rules:

1. For each propagator,

$$\begin{array}{c} \rightarrow \\ \bullet \text{---} \bullet \\ p \end{array} = \frac{i}{p^2 - m^2 + i\epsilon}$$

2. For each vertex,

$$\begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ z \end{array} = (-i\lambda)$$

3. For each external point,

$$\begin{array}{c} \leftarrow \\ \bullet \text{---} \\ p \end{array} = 1$$

4. Impose momentum conservation at each vertex.
5. Integrate over each undetermined momentum;

$$\int \frac{d^4 p}{(2\pi)^4}$$

6. Divide by the symmetry factor S .

6 Cross sections and decay rates

We have learned how to compute an abstract quantity, – the n -point correlation function –, through Feynman diagrams. We now want to relate this quantity to quantities, which can be measured: Cross sections and decay rates.

Elementary particle experiments are often scattering experiments: One collides two beams of particles with well-defined momenta, and observes what comes out. The likelihood of any particular final state can be expressed in terms of the cross section. This is a quantity intrinsic to the colliding particles and therefore allows comparison of two different experiments with different beam sizes and intensities.

The cross section is defined as follows: Consider a target at rest of particles of type A with density ρ_A (particles per unit volume). Aim at this target a bunch of particles of type B with number density ρ_B and velocity v . Let l_A and l_B be the lengths of the bunches of the particles and F the cross-sectional area common to the two bunches. We expect the total number of scattering events to be proportional to ρ_A , l_A , ρ_B , l_B and F . The cross section, denoted by σ , is just the total number of events divided by all these quantities:

$$\sigma = \frac{\text{Number of scattering events}}{\rho_A l_A \rho_B l_B F}.$$

Obviously, this definition is symmetric between A and B , therefore we could have worked in any other reference frame.

The cross section has units of area.

In real beams, ρ_A and ρ_B are not constant, the particle density is generally larger at the center of the beam than at the edges and one obtains

$$\text{Number of scattering events} = \sigma l_A l_B \int d^2x \rho_A(x) \rho_B(x).$$

If the densities are constant, we simply have

$$\text{Number of scattering events} = \frac{\sigma N_A N_B}{F},$$

where N_A and N_B are the total number of A and B particles.

Cross sections for many different processes may be relevant to a single scattering experiment. In e^+e^- collisions, for example, one can measure the cross section for production of $\mu^+\mu^-$, $\tau^+\tau^-$, $\mu^+\mu^-\gamma$, etc..

Usually we wish to measure not only what the final-state particles are, but also the momenta with which they come out. For that we define the differential cross section

$$\frac{d\sigma}{d^3p_1 \dots d^3p_n}$$

which is the quantity when integrated over any small $d^3 p_1 \dots d^3 p_n$ gives the cross section for scattering into that region of final-state phase space. Note that the various final-state momenta are not all independent: Four components will always be constrained by momentum conservation. In the simplest case of only two final-state particles this leaves only two unconstrained variables, usually taken to be the angles θ and ϕ of the momentum of one of the particles. Integrating

$$\frac{d\sigma}{d^3 p_1 d^3 p_2}$$

over the four constrained momentum components then leaves us with the usual differential cross section

$$\frac{d\sigma}{d\Omega}.$$

A second measurable quantity is the decay rate of an unstable particle. The decay rate Γ of an unstable particle A , which is assumed to be at rest, is defined as follows:

$$\Gamma = \frac{\text{Number of decays per unit time}}{\text{Number of } A \text{ particles present}}.$$

The lifetime τ is the reciprocal of the sum of the decay widths into all possible final states. The particle's half-life time is $\tau \cdot \ln 2$.

In non-relativistic quantum mechanics, an unstable atomic state shows up in scattering experiments as a resonance. Near the resonance energy E_0 , the scattering amplitude is given by the Breit-Wigner-formula

$$f(E) \sim \frac{1}{E - E_0 + i\Gamma/2}$$

The cross section therefore has a peak of the form

$$\sigma \sim \frac{1}{(E - E_0)^2 + \Gamma^2/4}$$

The width of the resonance peak is equal to the decay rate of the unstable state. The Breit-Wigner formula also applies to relativistic quantum mechanics. In an amplitude involving an unstable particle of momentum p and mass m we have

$$\frac{1}{p^2 - m^2 + im\Gamma} \approx \frac{1}{2E_{\vec{p}} (p^0 - E_{\vec{p}} + i(m/E_{\vec{p}})\Gamma/2)}.$$

Remark: A correct treatment of unstable particles in higher orders in perturbation theory is a topic of current interest.

6.1 The S-matrix

To calculate the cross section we set up wavepackets representing the initial-state particles, evolve this initial state for a very long time with the time-evolution operator $\exp(-iHt)$ of the interacting field theory and overlap the resulting final state with wavepackets representing some desired set of final-state particles.

A wavepacket is given by

$$|\phi\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} \phi(\vec{k}) |\vec{k}\rangle,$$

where $\phi(\vec{k})$ is the Fourier transform of the spatial wave function, and $|\vec{k}\rangle$ is a one-particle state of momentum \vec{k} in the interacting theory.

Normalisation;

$$\int \frac{d^3k}{(2\pi)^3} |\phi(\vec{k})|^2 = 1, \quad \langle\phi|\phi\rangle = 1.$$

The probability we wish to compute is

$$P = \left| \left\langle \underbrace{\phi_1\phi_2\dots}_{future} \middle| \underbrace{\phi_A\phi_B}_{past} \right\rangle \right|^2$$

These states are called in- and out-states. These states are in the Heisenberg picture, therefore time-independent, but the name we give a state depends on the eigenvalues or expectation values of time-dependent operators. A simple way to see this is the connection with the Schrödinger picture:

$$|\phi(t)\rangle_S = e^{-iH(t-t_0)} |\phi\rangle_H.$$

Let us consider the in-states: Consider particle A at rest and particle B along the z -axis. It is important to take the transverse displacement into account. If the wave packet of particle B is not centered at $x = y = 0$ in the transverse plane, but at \vec{b} , this yields a factor $\exp(-i\vec{b}\cdot\vec{k}_B)$. $|\vec{b}|$ is called the impact parameter.

$$|\phi_A\phi_B\rangle_{in} = \int \frac{d^3k_A}{(2\pi)^3} \int \frac{d^3k_B}{(2\pi)^3} \frac{\phi_A(\vec{k}_A)\phi_B(\vec{k}_B)}{\sqrt{(2E_{\vec{k}_A})(2E_{\vec{k}_B})}} e^{-i\vec{b}\cdot\vec{k}_B} |\vec{k}_A\vec{k}_B\rangle_{in}.$$

The wave packet of particle A is centered around p_A in momentum space, the one of particle B is centered around p_B . Similar,

$${}_{out} \langle\phi_1\phi_2\dots| = \left(\prod_f \int \frac{d^3q_f}{(2\pi)^3} \frac{\phi_f(\vec{q}_f)}{\sqrt{2E_{\vec{q}_f}}} \right) {}_{out} \langle\vec{q}_1\vec{q}_2\dots|.$$

Again, the wave packet for particle f is centered around p_f in momentum space. We have

$$\begin{aligned} {}_{out}\langle \vec{q}_1 \vec{q}_2 \dots | \vec{k}_A \vec{k}_B \rangle_{in} &= \lim_{T \rightarrow \infty} \left\langle \underbrace{\vec{q}_1 \vec{q}_2 \dots}_T \left| \underbrace{\vec{k}_A \vec{k}_B}_{-T} \right. \right\rangle \\ &= \lim_{T \rightarrow \infty} \left\langle \underbrace{\vec{q}_1 \vec{q}_2 \dots}_{t_0} \left| e^{-iH(2T)} \right| \underbrace{\vec{k}_A \vec{k}_B}_{t_0} \right\rangle \end{aligned}$$

We define the S -matrix by

$${}_{out}\langle \vec{q}_1 \vec{q}_2 \dots | \vec{k}_A \vec{k}_B \rangle_{in} = \left\langle \underbrace{\vec{q}_1 \vec{q}_2 \dots}_{t_0} \left| S \right| \underbrace{\vec{k}_A \vec{k}_B}_{t_0} \right\rangle$$

If the particles do not interact at all, S is the identity operator. Even if the theory contains interactions, the particles have some probability of missing each other. To isolate the interesting part of the S -matrix – that is, the part due to interactions – we define the T -matrix by

$$S = \mathbf{1} + i(2\pi)^4 \delta^4(k_A + k_B - \sum q_f) T.$$

Of special interest are the matrix elements of T :

$$i\mathcal{M}(k_A k_B \rightarrow q_1 q_2 \dots) = \langle \vec{q}_1 \vec{q}_2 \dots | iT | \vec{k}_A \vec{k}_B \rangle$$

The probability we want to compute is now

$$P = \left(\prod_f \int \frac{d^3 q_f}{(2\pi)^3} \frac{|\phi_f(\vec{q}_f)|^2}{2E_{\vec{q}_f}} \right) |{}_{out}\langle \vec{q}_1 \vec{q}_2 \dots | \phi_A \phi_B \rangle_{in}|^2$$

The in-state still depends on the impact parameter \vec{b} . To obtain the cross section we have to integrate over all impact parameters:

$$\sigma = \int d^2 b P(\vec{b}).$$

Therefore

$$\begin{aligned} \frac{d\sigma}{\frac{d^3 p_1}{(2\pi)^3} \frac{d^3 p_2}{(2\pi)^3} \dots} &= \left(\prod_f \int \frac{d^3 q_f}{(2\pi)^3} \frac{|\phi_f(\vec{q}_f)|^2}{2E_{\vec{q}_f}} \right) \int d^2 b \left(\prod_{i=A,B} \int \frac{d^3 k_i}{(2\pi)^3} \frac{\phi_i(\vec{k}_i)}{\sqrt{2E_{\vec{k}_i}}} \int \frac{d^3 k'_i}{(2\pi)^3} \frac{\phi_i^*(\vec{k}'_i)}{\sqrt{2E_{\vec{k}'_i}}} \right) \\ &e^{-i\vec{b} \cdot (\vec{k}_B - \vec{k}'_B)} {}_{out}\langle \vec{q}_1 \vec{q}_2 \dots | k_A k_B \rangle_{in} {}_{out}\langle \vec{q}_1 \vec{q}_2 \dots | k'_A k'_B \rangle_{in}^* \end{aligned}$$

If we are not interested in the trivial case where no scattering case takes place, we can drop the identity from the S -matrix and write

$$\begin{aligned} {}_{out}\langle \vec{q}_1 \vec{q}_2 \dots | k_A k_B \rangle_{in} &= (2\pi)^4 \delta^4(k_A + k_B - \sum q_f) i\mathcal{M}(k_A k_B \rightarrow q_1 q_2 \dots), \\ {}_{out}\langle \vec{q}_1 \vec{q}_2 \dots | k'_A k'_B \rangle_{in} &= (2\pi)^4 \delta^4(k'_A + k'_B - \sum q_f) i\mathcal{M}(k'_A k'_B \rightarrow q_1 q_2 \dots). \end{aligned}$$

We can use the second of these delta functions, together with the $\delta^2(k_B^\perp - k_B^{\perp'})$ to perform all six k' integrals. We have

$$\begin{aligned}
& \int \frac{d^3 k'_A}{(2\pi)^3} \int \frac{d^3 k'_B}{(2\pi)^3} (2\pi)^2 \delta^2(k_B^\perp - k_B^{\perp'}) (2\pi)^4 \delta^4(k'_A + k'_B - k_A - k_B) = \\
& = \int dk_A^{z'} \int dk_B^{z'} \delta(E'_A + E'_B - E_A - E_B) \delta(k_A^{z'} + k_B^{z'} - k_A^z - k_B^z) \\
& = \int dk_A^{z'} \delta(E'_A + E'_B - E_A - E_B) \Big|_{k_B^{z'} = k_A^z + k_B^z - k_A^{z'}} \\
& = \frac{1}{\left| \frac{k_A^{z'}}{E'_A} - \frac{k_B^z}{E_B} \right|} = \frac{1}{|v_A - v_B|}.
\end{aligned}$$

The difference $|v_A - v_B|$ is the relative velocity of the beams in the lab frame. The initial wave packets are localized in momentum space, centered around p_A and p_B . We evaluate all factors, which are smooth functions of k_A and k_B at p_A and p_B . We obtain

$$\begin{aligned}
\frac{d\sigma}{\frac{d^3 p_1}{(2\pi)^3} \frac{d^3 p_2}{(2\pi)^3} \dots} & = \left(\prod_f \frac{1}{2E_{\vec{p}_f}} \right) \left(\prod_f \int \frac{d^3 q_f}{(2\pi)^3} |\phi_f(\vec{q}_f)|^2 \right) \frac{|\mathcal{M}(p_A p_B \rightarrow p_1 p_2 \dots)|^2}{2E_A 2E_B |v_A - v_B|} \\
& \int \frac{d^3 k_A}{(2\pi)^3} \int \frac{d^3 k_B}{(2\pi)^3} |\phi_A(\vec{k}_A)|^2 |\phi_B(\vec{k}_B)|^2 (2\pi)^4 \delta^4(k_A + k_B - \sum q_f)
\end{aligned}$$

We can simplify this formula further by noting that real detectors cannot resolve small variations of the incoming and outgoing momenta. Therefore we can approximate $\delta(k_A + k_B - \sum q_f)$ by $\delta(p_A + p_B - \sum p_f)$ and we obtain

$$\frac{d\sigma}{\frac{d^3 p_1}{(2\pi)^3} \frac{d^3 p_2}{(2\pi)^3} \dots} = \left(\prod_f \frac{1}{2E_{\vec{p}_f}} \right) \frac{|\mathcal{M}(p_A p_B \rightarrow p_1 p_2 \dots)|^2}{2E_A 2E_B |v_A - v_B|} (2\pi)^4 \delta^4(p_A + p_B - \sum p_f)$$

Note that all dependence on the shape of the wave packets has disappeared.

By a similar reasoning we find for the decay rate

$$\frac{d\Gamma}{\frac{d^3 p_1}{(2\pi)^3} \frac{d^3 p_2}{(2\pi)^3} \dots} = \left(\prod_f \frac{1}{2E_{\vec{p}_f}} \right) \frac{|\mathcal{M}(p_A \rightarrow p_1 p_2 \dots)|^2}{2m_A} (2\pi)^4 \delta^4(p_A - \sum p_f)$$

To obtain the total cross section or the total decay rate we integrate over all final states. If the final state contains n identical particles, we have to be careful not to count the same final state more than once. One possibility is to impose a strict ordering on the energies of the final state particles

$$E_1 > E_2 > \dots > E_n,$$

a different (and in practice the preferred option) is to integrate without any restriction and to divide the result by a factor of $1/n!$. Therefore

$$\begin{aligned}\sigma &= \frac{1}{2E_A 2E_B |v_A - v_B|} \frac{1}{n!} \int \left(\prod_f \frac{d^3 p_f}{(2\pi)^3 2E_{\vec{p}_f}} \right) (2\pi)^4 \delta^4(p_A + p_B - \sum p_f) |\mathcal{M}(p_A p_B \rightarrow p_1 p_2 \dots)|^2, \\ \Gamma &= \frac{1}{2m_A} \frac{1}{n!} \int \left(\prod_f \frac{d^3 p_f}{(2\pi)^3 2E_{\vec{p}_f}} \right) (2\pi)^4 \delta^4(p_A - \sum p_f) |\mathcal{M}(p_A \rightarrow p_1 p_2 \dots)|^2,\end{aligned}$$

We denote the Lorentz-invariant phase-space element by

$$\begin{aligned}d\phi(Q; p_1, p_2, \dots, p_n) &= \frac{1}{n!} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3 2E_{\vec{p}_f}} \right) (2\pi)^4 \delta^4(Q - \sum p_f) \\ &= \frac{1}{n!} \left(\prod_f \frac{d^4 p_f}{(2\pi)^4} (2\pi) \delta(p_f^2 - m^2) \theta(E_{\vec{p}_f}) \right) (2\pi)^4 \delta^4(Q - \sum p_f)\end{aligned}$$

We can also bring the flux factor into a Lorentz-invariant form:

$$\frac{1}{2E_A 2E_B |v_A - v_B|} = \frac{1}{2K(Q^2)},$$

where

$$\begin{aligned}2K(Q^2) &= 2\sqrt{(Q^2 - (m_1 + m_2)^2)(Q^2 - (m_1 - m_2)^2)} \\ &= 2Q^2 \quad \text{for massless particles,}\end{aligned}$$

and $Q^2 = (p_A + p_B)^2$. Finally, if the initial-state particles have additional degrees of freedom, like spin, we have to average over these degrees of freedom. Since we sum over all spins in

$$|\mathcal{M}|^2,$$

this brings a an additional factor of

$$\frac{1}{(2J_1 + 1)} \frac{1}{(2J_2 + 1)}$$

for scattering reactions and a factor

$$\frac{1}{(2J_1 + 1)}$$

for the decay rate.

6.2 Relation between invariant matrix elements and Feynman diagrams

Here we sketch only a heuristic argument for the relation of the S-matrix to Feynman diagrams and postpone the proof to later. The proof is based on the reduction formula of Lehmann, Symanzik and Zimmermann. The S-matrix was defined by

$$\langle \vec{p}_1 \vec{p}_2 \dots | S | \vec{p}_A \vec{p}_B \rangle = \lim_{T \rightarrow \infty} \langle \vec{p}_1 \vec{p}_2 \dots | e^{-iH(2T)} | \vec{p}_A \vec{p}_B \rangle$$

To compute this quantity we would like to replace the external plane-wave states, which are eigenstates of H , with their counterparts in the free theory, which are eigenstates of H_0 . For the vacuum state we already found such a formula

$$|\Omega\rangle = \lim_{T \rightarrow \infty (1-i\epsilon)} \left(e^{-i\hat{E}_0 T} \langle \Omega | 0 \rangle \right)^{-1} e^{-iHT} |0\rangle$$

Now we look for a relation of the form

$$|\vec{p}_A \vec{p}_B\rangle \sim \lim_{T \rightarrow \infty (1-i\epsilon)} e^{-iHT} |\vec{p}_A \vec{p}_B\rangle_0,$$

where we have omitted the prefactor. If this formula could be justified, we could then write

$$\lim_{T \rightarrow \infty (1-i\epsilon)} \langle \vec{p}_1 \vec{p}_2 \dots | e^{-iH(2T)} | \vec{p}_A \vec{p}_B \rangle \sim \lim_{T \rightarrow \infty (1-i\epsilon)} {}_0 \langle \vec{p}_1 \vec{p}_2 \dots | T \exp \left[-i \int_{-T}^T dt H_I(t) \right] | \vec{p}_A \vec{p}_B \rangle_0$$

In the case of vacuum expectation values, proportionality factors cancelled out in the final formula. It can be shown that the non-trivial part of the S-matrix can be computed as follows

$$\langle \vec{p}_1 \vec{p}_2 \dots | iT | \vec{p}_A \vec{p}_B \rangle = \lim_{T \rightarrow \infty (1-i\epsilon)} \left({}_0 \langle \vec{p}_1 \vec{p}_2 \dots | T \exp \left[-i \int_{-T}^T dt H_I(t) \right] | \vec{p}_A \vec{p}_B \rangle_0 \right)_{\text{connected, amputated}}$$

A Feynman diagram is called connected, if it does not have disjoint pieces.

A Feynman diagram is called amputated, if for all external lines, the corresponding propagator is removed.

6.3 Final formula

The cross section is given by To calculate an observable at an collider with no initial-state hadrons (e.g. an electron-positron collider):

$$\sigma = \frac{1}{2K(Q^2)} \frac{1}{(2J_1 + 1)} \frac{1}{(2J_2 + 1)} \int d\phi(p_A + p_B; p_1, \dots, p_n) |\mathcal{M}(p_A p_B \rightarrow p_1 p_2 \dots)|^2$$

where $2K(Q^2)$ is the flux factor and we have $2K(Q^2) = 2Q^2$ for massless incoming particles. For a decay rate we have

$$\Gamma = \frac{1}{2m_A} \frac{1}{(2J+1)} \int d\phi(p_A; p_1, \dots, p_n) \sum_{\text{helicity}} |\mathcal{M}(p_A \rightarrow p_1 p_2 \dots)|^2$$

The phase-space measure is given by

$$d\phi(Q; p_1, p_2, \dots, p_n) = \frac{1}{\prod_j n_j!} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3 2E_{\vec{p}_f}} \right) (2\pi)^4 \delta^4(Q - \sum p_f)$$

if the final state contains n_j identical particles of type j . If the colliding particles are not elementary (like protons or antiprotons), we have to include the probability of finding the elementary particle A inside the proton or antiproton. If the proton has momentum \hat{p}_p one usually specifies the probability of finding a parton with momentum fraction x by the parton distribution function

$$f(x)$$

The parton has then the momentum

$$p_A = x p_p.$$

For the cross section we have to integrate over all possible momentum fractions and the formula for a hadron-hadron collider becomes

$$\sigma = \int dx_1 f(x_1) \int dx_2 f(x_2) \frac{1}{2K(s)} \frac{1}{(2J_1+1)} \frac{1}{(2J_2+1)} \frac{1}{n_1 n_2} \int d\phi(p_A + p_B; p_1, \dots, p_n) |\mathcal{M}(p_A p_B \rightarrow p_1 p_2 \dots)|^2.$$

n_1 and n_2 are the number of colour degrees of the initial state particles.

7 Fermions

Up to now we considered only spin zero particles. In this section we study quantum field theories with spin 1/2 particles. But before we embark on quantization of these theories, we first discuss solutions of the classical theory.

7.1 The Dirac equation

The Lagrange density for a (classical) Dirac field depends on four-component spinors $\psi_\alpha(x)$ ($\alpha = 1, 2, 3, 4$) and $\bar{\psi}_\alpha(x) = (\psi^\dagger(x)\gamma^0)_\alpha$:

$$\mathcal{L}(\psi, \bar{\psi}, \partial_\mu \psi) = i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - m\bar{\psi}(x)\psi(x)$$

Here, the (4×4) -Dirac matrices satisfy the anti-commutation rules

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}, \quad \{\gamma^\mu, \gamma_5\} = 0, \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{24}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma.$$

The Euler-Lagrange equations yield the Dirac equations

$$\begin{aligned} (i\gamma^\mu\partial_\mu - m)\psi(x) &= 0, \\ \bar{\psi}(x)\left(i\overleftarrow{\gamma}^\mu\partial_\mu + m\right) &= 0. \end{aligned}$$

For computations it is useful to have an explicit representation of the Dirac matrices. There are several widely used representations. A particular useful one is the Weyl representation of the Dirac matrices:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Here, the 4-dimensional σ^μ -matrices are defined by

$$\sigma_{A\dot{B}}^\mu = (1, -\vec{\sigma}) \quad \bar{\sigma}^{\mu\dot{A}B} = (1, \vec{\sigma})$$

and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the standard Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let us now look for plane wave solutions of the Dirac equation. We make the Ansatz

$$\psi(x) = \begin{cases} u(p)e^{-ipx}, & p^0 > 0, \quad p^2 = m^2, \\ v(p)e^{+ipx}, & p^0 > 0, \quad p^2 = m^2. \end{cases}$$

$u(p)$ describes positive energy solutions, $v(p)$ describes negative energy solutions. Similar,

$$\bar{\psi}(x) = \begin{cases} \bar{u}(p)e^{+ipx}, & p^0 > 0, \quad p^2 = m^2, \\ \bar{v}(p)e^{-ipx}, & p^0 > 0, \quad p^2 = m^2, \end{cases}$$

where

$$\bar{u}(p) = u^\dagger(p)\gamma^0, \quad \bar{v}(p) = v^\dagger(p)\gamma^0.$$

Then

$$\begin{aligned} (\not{p}' - m)u(p) &= 0, & (\not{p}' + m)v(p) &= 0, \\ \bar{u}(p)(\not{p}' - m) &= 0, & \bar{v}(p)(\not{p}' + m) &= 0, \end{aligned}$$

We will find that there is more than one solution for $u(p)$ (and the other spinors $\bar{u}(p)$, $v(p)$, $\bar{v}(p)$). We will label the various solutions with λ . The degeneracy is related to the additional spin degree of freedom and we find for a spin 1/2 particles 2 solutions. We require that the two solutions satisfy the orthogonality relations

$$\begin{aligned} \bar{u}(p, \bar{\lambda})u(p, \lambda) &= 2m\delta_{\bar{\lambda}\lambda}, \\ \bar{v}(p, \bar{\lambda})v(p, \lambda) &= -2m\delta_{\bar{\lambda}\lambda}, \\ \bar{u}(p, \bar{\lambda})v(p, \lambda) &= \bar{v}(\bar{\lambda})u(\lambda) = 0, \end{aligned}$$

and the completeness relation

$$\begin{aligned} \sum_{\lambda} u(p, \lambda)\bar{u}(p, \lambda) &= \not{p}' + m, \\ \sum_{\lambda} v(p, \lambda)\bar{v}(p, \lambda) &= \not{p}' - m. \end{aligned}$$

7.2 Massless spinors

Let us now try to find explicit solutions for the spinors $u(p)$, $v(p)$, $\bar{u}(p)$ and $\bar{v}(p)$. The simplest case is the one of a massless fermion:

$$m = 0.$$

In this case the Dirac equation for the u - and the v -spinors are identical and it is sufficient to consider

$$\not{p}'u(p) = 0, \quad \bar{u}(p)\not{p}' = 0.$$

In the Weyl representation \not{p}' is given by

$$\not{p}' = \begin{pmatrix} 0 & p_\mu\sigma^\mu \\ p_\mu\bar{\sigma}^\mu & 0 \end{pmatrix},$$

therefore the 4×4 -matrix equation for $u(p)$ (or $\bar{u}(p)$) decouples into two 2×2 -matrix equations. We introduce the following notation: Four-component Dirac spinors are constructed out of two Weyl spinors as follows:

$$u(p) = \begin{pmatrix} |p+\rangle \\ |p-\rangle \end{pmatrix} = \begin{pmatrix} p_A \\ p^{\dot{B}} \end{pmatrix} = \begin{pmatrix} u_+(p) \\ u_-(p) \end{pmatrix}.$$

Bra-spinors are given by

$$\bar{u}(p) = (\langle p-|, \langle p+|) = (p^A, p_B) = (\bar{u}_-(p), \bar{u}_+(p)),$$

If we define the helicity projection operators

$$P_+ = \frac{1}{2}(1 + \gamma_5) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$P_- = \frac{1}{2}(1 - \gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

then

$$u_{\pm}(p) = P_{\pm}u(p), \quad \bar{u}_{\pm}(p) = \bar{u}(p)P_{\mp}.$$

The two solutions of the Dirac equation

$$\not{p}u(p, \lambda) = 0$$

are then

$$u(p, +) = u_+(p), \quad u(p, -) = u_-(p).$$

We now have to solve

$$p_{\mu}\bar{\sigma}^{\mu}|p+\rangle = 0, \quad p_{\mu}\sigma^{\mu}|p-\rangle = 0,$$

$$\langle p+|p_{\mu}\bar{\sigma}^{\mu} = 0, \quad \langle p-|p_{\mu}\sigma^{\mu} = 0.$$

It is convenient to express the four-vector $p_{\mu} = (p_0, p_1, p_2, p_3)$ in terms of light-cone coordinates:

$$p_+ = p_0 + p_3, \quad p_- = p_0 - p_3, \quad p_{\perp} = p_1 + ip_2, \quad p_{\perp}^* = p_1 - ip_2.$$

Note that p_{\perp}^* does not involve a complex conjugation of p_1 or p_2 . For null-vectors one has

$$p_{\perp}^*p_{\perp} = p_+p_-$$

Then the equation for the ket-spinors becomes

$$\begin{pmatrix} p_+ & p_{\perp}^* \\ p_{\perp} & p_- \end{pmatrix}|p+\rangle = 0, \quad \begin{pmatrix} p_- & -p_{\perp}^* \\ -p_{\perp} & p_+ \end{pmatrix}|p-\rangle = 0,$$

Solutions for ket-spinors are

$$|p+\rangle = c_+ e^{i\alpha_+} \begin{pmatrix} -p_{\perp}^* \\ p_+ \end{pmatrix}, \quad |p-\rangle = c_- e^{-i\alpha_-} \begin{pmatrix} p_+ \\ p_{\perp} \end{pmatrix}.$$

Solutions for bra-spinors are

$$\langle p+| = c_+ e^{-i\alpha'_+} (-p_\perp, p_+), \quad \langle p-| = c_- e^{i\alpha'_-} (p_+, p_\perp^*).$$

c_+ and c_- are real positive numbers. We assumed that $|p+\rangle$ and $\langle p+|$ have the same normalization factor c_+ , and a similar assumption was made for $|p-\rangle$ and $\langle p-|$. We normalize the massless spinors such that

$$\langle p\pm | \gamma_\mu | p\pm \rangle = 2p_\mu.$$

On the l.h.s we obtain

$$\begin{aligned} \langle p+ | \gamma_\mu | p+ \rangle &= \langle p+ | \bar{\sigma}_\mu | p+ \rangle = c_+^2 e^{i(\alpha_+ - \alpha'_+)} p_+ 2p_\mu, \\ \langle p- | \gamma_\mu | p- \rangle &= \langle p- | \sigma_\mu | p- \rangle = c_-^2 e^{-i(\alpha_- - \alpha'_-)} p_+ 2p_\mu. \end{aligned}$$

Since we assumed $p_0 > 0$ we have $p_+ \geq 0$. It follows that the constants c_+ and c_- are given by

$$c_+ = c_- = \frac{1}{\sqrt{|p_+|}}.$$

For the phases one has

$$\alpha'_+ = \alpha_+ \bmod 2\pi, \quad \alpha'_- = \alpha_- \bmod 2\pi.$$

In the following we will use

$$\alpha_+ = \alpha'_+ = \alpha_- = \alpha'_- = 0.$$

Therefore

$$\begin{aligned} |p+\rangle = p_A &= \frac{1}{\sqrt{|p_+|}} \begin{pmatrix} -p_\perp^* \\ p_+ \end{pmatrix}, & |p-\rangle = p^{\dot{B}} &= \frac{1}{\sqrt{|p_+|}} \begin{pmatrix} p_+ \\ p_\perp \end{pmatrix}, \\ \langle p-| = p^A &= \frac{1}{\sqrt{|p_+|}} (p_+, p_\perp^*), & \langle p+| = p_{\dot{B}} &= \frac{1}{\sqrt{|p_+|}} (-p_\perp, p_+). \end{aligned}$$

Let us now introduce the 2-dimensional antisymmetric tensor:

$$\varepsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon_{BA} = -\varepsilon_{AB}$$

Furthermore:

$$\varepsilon^{AB} = \varepsilon^{\dot{A}\dot{B}} = \varepsilon_{AB} = \varepsilon_{\dot{A}\dot{B}}.$$

We then find the following relations for raising and lowering a spinor index A or \dot{B} :

$$\begin{aligned} p^A &= \varepsilon^{AB} p_B, & p^{\dot{A}} &= \varepsilon^{\dot{A}\dot{B}} p_{\dot{B}}, \\ p_B &= p^{\dot{A}} \varepsilon_{\dot{A}B}, & p_{\dot{B}} &= p^A \varepsilon_{A\dot{B}}. \end{aligned}$$

Note that raising an index is done by left-multiplication, whereas lowering is performed by right-multiplication. Therefore the relation between the different spinors is as follows:

$$\begin{array}{ccc}
\langle p- | = p^A & \xleftarrow{\varepsilon^{AB}} & |p+\rangle = p_A \\
\uparrow \text{hermitian conj.} & & \uparrow \text{hermitian conj.} \\
|p-\rangle = p^{\dot{A}} & \xleftarrow{\varepsilon^{\dot{A}\dot{B}}} & \langle p+ | = p_{\dot{A}}
\end{array}$$

7.3 Spinorproducts

We can define two spinorproducts

$$\begin{aligned}
\langle pq \rangle &= \langle p- | q+ \rangle = p^A q_A = \frac{1}{\sqrt{|p+||q+|}} (p_{\perp}^* q_+ - p_+ q_{\perp}^*), \\
[qp] &= \langle q+ | p- \rangle = q_{\dot{B}} p^{\dot{B}} = \frac{1}{\sqrt{|p+||q+|}} (p_{\perp} q_+ - p_+ q_{\perp}).
\end{aligned}$$

We have

$$\langle pq \rangle [qp] = 2p \cdot q.$$

Therefore

$$|\langle pq \rangle| = |[qp]| = \sqrt{2p \cdot q}.$$

The spinorproducts are anti-symmetric:

$$\begin{aligned}
\langle qp \rangle &= -\langle pq \rangle, \\
[qp] &= -[qp].
\end{aligned}$$

From the Schouten identity for the 2-dimensional antisymmetric tensor

$$\varepsilon_{AB}\varepsilon_{CD} + \varepsilon_{AC}\varepsilon_{DB} + \varepsilon_{AD}\varepsilon_{BC} = 0$$

one derives

$$\begin{aligned}
\langle AB \rangle \langle CD \rangle &= \langle AD \rangle \langle CB \rangle + \langle AC \rangle \langle BD \rangle \\
[AB] [CD] &= [AD] [CB] + [AC] [BD]
\end{aligned}$$

Fierz identity:

$$\langle A + |\gamma_{\mu} B \rangle \langle C - |\gamma^{\mu} D \rangle = 2[AD] \langle CB \rangle$$

Useful formulas in the bra-ket notation:

$$\begin{aligned}
\langle p \pm |\gamma_{\mu_1} \dots \gamma_{\mu_{2n+1}} | q \pm \rangle &= \langle q \mp |\gamma_{\mu_{2n+1}} \dots \gamma_{\mu_1} | p \mp \rangle \\
\langle p \pm |\gamma_{\mu_1} \dots \gamma_{\mu_{2n}} | q \mp \rangle &= -\langle q \pm |\gamma_{\mu_{2n}} \dots \gamma_{\mu_1} | p \mp \rangle
\end{aligned}$$

7.4 Massive spinors

As in the massless case, a massive spinor satisfying the Dirac equation has a two-fold degeneracy. We will label the two different eigenvectors by “+” and “-”. Let p be a massive four-vector with $p^2 = m^2$, and let q be an arbitrary light-like vector. With the help of q we can construct a light-like vector p^b associated to p :

$$p^b = p - \frac{p^2}{2p \cdot q} q.$$

We define

$$\begin{aligned} u(p, +) &= \frac{1}{\langle p^b + | q - \rangle} (\not{p}' + m) | q - \rangle, & v(p, +) &= \frac{1}{\langle p^b + | q - \rangle} (\not{p}' - m) | q - \rangle, \\ u(p, -) &= \frac{1}{\langle p^b - | q + \rangle} (\not{p}' + m) | q + \rangle, & v(p, -) &= \frac{1}{\langle p^b - | q + \rangle} (\not{p}' - m) | q + \rangle. \end{aligned}$$

For the conjugate spinors we have

$$\begin{aligned} \bar{u}(p, +) &= \frac{1}{\langle q - | p^b + \rangle} \langle q - | (\not{p}' + m), & \bar{v}(p, +) &= \frac{1}{\langle q - | p^b + \rangle} \langle q - | (\not{p}' - m), \\ \bar{u}(p, -) &= \frac{1}{\langle q + | p^b - \rangle} \langle q + | (\not{p}' + m), & \bar{v}(p, -) &= \frac{1}{\langle q + | p^b - \rangle} \langle q + | (\not{p}' - m). \end{aligned}$$

These spinors satisfy the Dirac equations

$$\begin{aligned} (\not{p}' - m) u(\lambda) &= 0, & (\not{p}' + m) v(\lambda) &= 0, \\ \bar{u}(\lambda) (\not{p}' - m) &= 0, & \bar{v}(\lambda) (\not{p}' + m) &= 0, \end{aligned}$$

the orthogonality relations

$$\begin{aligned} \bar{u}(\tilde{\lambda}) u(\lambda) &= 2m \delta_{\tilde{\lambda}\lambda}, \\ \bar{v}(\tilde{\lambda}) v(\lambda) &= -2m \delta_{\tilde{\lambda}\lambda}, \\ \bar{u}(\tilde{\lambda}) v(\lambda) &= \bar{v}(\tilde{\lambda}) u(\lambda) = 0, \end{aligned}$$

and the completeness relation

$$\begin{aligned} \sum_{\lambda} u(\lambda) \bar{u}(\lambda) &= \not{p}' + m, \\ \sum_{\lambda} v(\lambda) \bar{v}(\lambda) &= \not{p}' - m. \end{aligned}$$

7.5 Quantization of fermions

We start from the Lagrangian

$$\mathcal{L}(\psi, \bar{\psi}, \partial_{\mu}\psi) = i\bar{\psi}(x)\gamma^{\mu}\partial_{\mu}\psi(x) - m\bar{\psi}(x)\psi(x)$$

The canonical momentum conjugate to ψ is

$$\frac{\partial \mathcal{L}}{\partial \partial_0 \psi} = i\bar{\psi}\gamma^0 = i\psi^\dagger.$$

This the Hamiltonian is

$$H = \int d^3x i\psi^\dagger \partial_0 \psi - \mathcal{L} = \int d^3x \bar{\psi} \left[-i\vec{\gamma} \cdot \vec{\nabla} + m \right] \psi = \int d^3x \psi^\dagger \left[-i\gamma^0 \vec{\gamma} \cdot \vec{\nabla} + m\gamma^0 \right] \psi.$$

Here

$$\vec{\gamma} = (\gamma^1, \gamma^2, \gamma^3), \quad \vec{\nabla} = (\partial_1, \partial_2, \partial_3).$$

If we define

$$\vec{\alpha} = \gamma^0 \vec{\gamma}, \quad \beta = \gamma^0,$$

we obtain

$$H = \int d^3x \psi^\dagger \left[-i\vec{\alpha} \cdot \vec{\nabla} + m\beta \right] \psi.$$

Let us expand the field $\psi(x)$ in a set of eigenfunctions of

$$h_D = -i\vec{\alpha} \cdot \vec{\nabla} + m\beta$$

From the solution of the free Dirac equation we already know that

$$\left[-i\vec{\alpha} \cdot \vec{\nabla} + m\beta \right] u(p, \lambda) e^{-ip \cdot x} = 0.$$

Therefore $u(p, \lambda) e^{i\vec{p} \cdot \vec{x}}$ are eigenfunctions of h_D with eigenvalues $E_{\vec{p}}$. Similarly, the functions $v(p, \lambda) e^{-i\vec{p} \cdot \vec{x}}$ are eigenfunctions of h_D with eigenvalues $-E_{\vec{p}}$. These form a complete set of eigenfunctions, since for any \vec{p} there are two u 's and two v 's, giving us four eigenvectors of the 4×4 matrix h_D . We write for the field operators in the Heisenberg picture

$$\begin{aligned} \psi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{\lambda} \left(a_{\vec{p}, \lambda} u(p, \lambda) e^{-ip \cdot x} + b_{\vec{p}, \lambda}^\dagger v(p, \lambda) e^{ip \cdot x} \right), \\ \bar{\psi}(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{\lambda} \left(b_{\vec{p}, \lambda} \bar{v}(p, \lambda) e^{-ip \cdot x} + a_{\vec{p}, \lambda}^\dagger \bar{u}(p, \lambda) e^{ip \cdot x} \right). \end{aligned}$$

The creation and annihilation operators obey the anticommutation rules

$$\left\{ a_{\vec{p}, \lambda}, a_{\vec{q}, \lambda'}^\dagger \right\} = \left\{ b_{\vec{p}, \lambda}, b_{\vec{q}, \lambda'}^\dagger \right\} = (2\pi)^3 \delta^3(\vec{p} - \vec{q}) \delta_{\lambda \lambda'}.$$

The equal-time anticommutation relations for ψ and ψ^\dagger are then

$$\begin{aligned}\{\psi(\vec{x}), \psi^\dagger(\vec{y})\} &= \delta^3(\vec{x} - \vec{y}), \\ \{\psi(\vec{x}), \psi(\vec{y})\} &= \{\psi^\dagger(\vec{x}), \psi^\dagger(\vec{y})\} = 0.\end{aligned}$$

The vacuum $|0\rangle$ is defined to be the state such

$$a_{\vec{p}, \lambda}|0\rangle = b_{\vec{p}, \lambda}|0\rangle = 0.$$

The Hamiltonian can be written as

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_{\lambda} E_{\vec{p}} \left(a_{\vec{p}, \lambda}^\dagger a_{\vec{p}, \lambda} + b_{\vec{p}, \lambda}^\dagger b_{\vec{p}, \lambda} \right),$$

where an infinite constant has been dropped. The momentum operator is

$$\vec{P} = \int d^3x \psi^\dagger (-i\vec{\nabla}) \psi = \int \frac{d^3p}{(2\pi)^3} \sum_{\lambda} \vec{p} \left(a_{\vec{p}, \lambda}^\dagger a_{\vec{p}, \lambda} + b_{\vec{p}, \lambda}^\dagger b_{\vec{p}, \lambda} \right)$$

Thus both $a_{\vec{p}, \lambda}^\dagger$ and $b_{\vec{p}, \lambda}^\dagger$ create particles with energy $+E_{\vec{p}}$ and momentum \vec{p} . We will refer to the particles created by $a_{\vec{p}, \lambda}^\dagger$ as fermions and to those created by $b_{\vec{p}, \lambda}^\dagger$ as antifermions.

The one-particle states

$$|\vec{p}, \lambda\rangle = \sqrt{2E_{\vec{p}}} a_{\vec{p}, \lambda}^\dagger |0\rangle$$

are defined so that their inner product

$$\langle \vec{p}, \lambda | \vec{q}, \lambda' \rangle = 2E_{\vec{p}} (2\pi)^3 \delta^3(\vec{p} - \vec{q}) \delta_{\lambda\lambda'}$$

is Lorentz invariant.

7.6 Feynman rules for fermions

To apply Wick's theorem we have to generalize the definitions of the time-ordered product and the normal product towards anticommuting operators. We make the following definitions:

$$T\psi(x)\bar{\psi}(y) = \begin{cases} \psi(x)\bar{\psi}(y) & \text{for } x^0 > y^0 \\ -\bar{\psi}(y)\psi(x) & \text{for } y^0 > x^0 \end{cases}$$

For the normal product we define

$$\begin{aligned}: a_{\vec{p}}^\dagger a_{\vec{q}} : &= a_{\vec{p}}^\dagger a_{\vec{q}}, \\ : a_{\vec{q}} a_{\vec{p}}^\dagger : &= -a_{\vec{p}}^\dagger a_{\vec{q}},\end{aligned}$$

where a_p^\dagger and $a_{\bar{q}}$ are creation and annihilation operators for fermions. Therefore a minus sign occurs everytime we have to exchange two fermionic operators.

The Lagrange density for the Yukawa theory:

$$\begin{aligned}\mathcal{L}_{Yukawa} &= \mathcal{L}_{Dirac} + \mathcal{L}_{Klein-Gordon} - g\bar{\psi}\psi\phi, \\ \mathcal{L}_{Dirac} &= i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi, \\ \mathcal{L}_{Klein-Gordon} &= \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2.\end{aligned}$$

We obtain the following Feynman rules in momentum space:

1. For each propagator,

$$\begin{aligned}\bullet \xrightarrow{p} \bullet &= \frac{i}{p^2 - m^2 + i\epsilon} \\ \bullet \xrightarrow{p} \bullet &= \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}\end{aligned}$$

2. For each vertex,

$$\begin{array}{c} \diagdown \\ \bullet \\ \diagup \end{array} = -ig$$

3. For each external point,

$$\begin{aligned}\bullet \xleftarrow{p} &= 1 \\ \bullet \xleftarrow{p} &= \bar{u}(p, \lambda) \\ \bullet \xleftarrow{p} &= v(p, \lambda) \\ \bullet \xrightarrow{p} &= \bar{v}(p, \lambda) \\ \bullet \xrightarrow{p} &= u(p, \lambda)\end{aligned}$$

4. Impose momentum conservation at each vertex.

5. Integrate over each undetermined momentum;

$$\int \frac{d^4p}{(2\pi)^4}$$

6. Divide by the symmetry factor S .

7. For each closed fermion loop a factor of (-1) .

7.7 Rules for traces over Dirac matrices

In evaluating the amplitude squared we encounter trace over Dirac matrices:

Theorem 1:

$$\text{Tr } \gamma^\mu \gamma^\nu = 4g^{\mu\nu}$$

Proof:

$$\text{Tr } \gamma^\mu \gamma^\nu + \text{Tr } \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \text{Tr } 1 = 8g^{\mu\nu}$$

Theorem 2:

$$\text{Tr } \gamma^{\mu_1} \gamma^{\mu_2} \dots \gamma^{\mu_{2n}} = g^{\mu_1 \mu_2} \text{Tr } \gamma^{\mu_3} \dots \gamma^{\mu_{2n}} - g^{\mu_1 \mu_3} \text{Tr } \gamma^{\mu_2} \gamma^{\mu_4} \dots \gamma^{\mu_{2n}} + g^{\mu_1 \mu_4} \text{Tr } \gamma^{\mu_2} \gamma^{\mu_3} \gamma^{\mu_5} \dots \gamma^{\mu_{2n}} - \dots$$

Proof:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \quad + \text{cyclic property of the trace}$$

Theorem 3:

$$\text{Tr } \gamma^{\mu_1} \gamma^{\mu_2} \dots \gamma^{\mu_{2n-1}} = 0$$

Proof:

$$\text{Tr } \gamma^\mu = \text{Tr } \gamma_5 \gamma_5 \gamma^\mu = -\text{Tr } \gamma_5 \gamma^\mu \gamma_5 = -\text{Tr } \gamma^\mu$$

Theorem 4:

$$\text{Tr } \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5 = 4i \epsilon^{\mu\nu\rho\sigma}.$$

8 Quantum field theory via path integrals

Review of the canonical formalism for the quantization of field theories: We are given a Lagrange density

$$\mathcal{L}(\phi, \partial_\mu \phi) = \mathcal{L}_0(\phi, \partial_\mu \phi) + \mathcal{L}_{int}(\phi, \partial_\mu \phi),$$

which can be split into a “free” part \mathcal{L}_0 (bilinear in the fields) and a part \mathcal{L}_{int} describing the interactions. Each term in \mathcal{L}_{int} contains at least three fields.

The the momentum density conjugate to $\phi(x)$ is given by

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)}.$$

The Hamiltonian is given by

$$H = \int d^3x \left[\pi(x) \dot{\phi}(x) - \mathcal{L} \right] = \int d^3x \mathcal{H}.$$

In the canonical formalism the field $\phi(x)$ and the conjugate momentum $\pi(x)$ become operators. In the Schrödinger picture the operators are time-independent. We postulate canonical (anti)-commutation relations. For bosons we require

$$\begin{aligned} [\phi(t_0, \vec{x}), \pi(t_0, \vec{y})] &= i\delta^3(\vec{x} - \vec{y}), \\ [\phi(t_0, \vec{x}), \phi(t_0, \vec{y})] &= [\pi(t_0, \vec{x}), \pi(t_0, \vec{y})] = 0. \end{aligned}$$

For fermions we have (recall that the conjugate momentum is $\partial \mathcal{L} / \partial (\partial_0 \psi) = i\psi^\dagger$):

$$\begin{aligned} \{ \psi(t_0, \vec{x}), i\psi^\dagger(t_0, \vec{y}) \} &= i\delta^3(\vec{x} - \vec{y}), \\ \{ \psi(t_0, \vec{x}), \psi(t_0, \vec{y}) \} &= \{ i\psi^\dagger(t_0, \vec{x}), i\psi^\dagger(t_0, \vec{y}) \} = 0. \end{aligned}$$

To change from the Schrödinger picture to the Heisenberg picture we have the formulae

$$\begin{aligned} \phi(t, \vec{x}) &= e^{iH(t-t_0)} \phi(t_0, \vec{x}) e^{-iH(t-t_0)}, \\ \pi(t, \vec{x}) &= e^{iH(t-t_0)} \pi(t_0, \vec{x}) e^{-iH(t-t_0)}. \end{aligned}$$

We can also split the Hamiltonian into a “free” part and a piece describing the interactions:

$$H = H_0 + H_{int}.$$

We define the field operator in the interaction picture as

$$\phi_I(t, \vec{x}) = e^{iH_0(t-t_0)} \phi(t_0, \vec{x}) e^{-iH_0(t-t_0)}.$$

We have the following relation between the field operators in the Heisenberg picture and the interaction picture:

$$\phi(x) = U^\dagger(t, t_0)\phi_I(x)U(t, t_0),$$

where

$$U(t, t') = T \left\{ \exp \left[-i \int_{t'}^t dt'' H_I(t'') \right] \right\}.$$

8.1 Trouble with the canonical quantization of gauge bosons

The canonical formalism worked fine for the quantization of spin 0 and spin 1/2 particles. Let us now consider spin 1 particles. As a first example we consider the photon field without any interactions with fermions. The Lagrange density is

$$\mathcal{L}(A_\mu, \partial_\mu A_\nu) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

The canonical momentum conjugate to A_μ is given by

$$\Pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}^\mu} = -F_{0\mu}.$$

The canonical commutation relation would be given by

$$[A_\mu(t_0, \vec{x}), \Pi_\nu(t_0, \vec{y})] = ig_{\mu\nu}\delta^3(\vec{x} - \vec{y}).$$

If we set $\mu = \nu = 0$ we obtain

$$[A_0(t_0, \vec{x}), \Pi_0(t_0, \vec{y})] = i\delta^3(\vec{x} - \vec{y}).$$

On the other hand, we have

$$\Pi_0(t_0, \vec{y}) = -F_{00} = 0.$$

Thus the simple-minded application of the canonical quantization fails. The problem is related to the invariance of the Lagrange density under the gauge transformation

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \Lambda(x).$$

One possibility to circumvent the problem is to eliminate the freedom of gauge transformations by putting constraints on the field A_μ . One adds a gauge-fixing term to the Lagrange density:

$$\mathcal{L}(A_\mu, \partial_\mu A_\nu) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

Remark: The Lagrange density is no longer gauge-invariant. The canonical momentum is now given by

$$\Pi_\mu = -F_{0\mu} - \frac{1}{\xi} g_{0\mu} (\partial^\nu A_\nu).$$

The above ad-hoc recipe works fine for abelian gauge theories like QED, but fails for non-abelian gauge theories, which are needed for the strong and weak interactions.

For a deeper understanding of the problem and its solution it is simpler to switch to a second method for the quantization of field theories: quantization in the path-integral formalism.

It should be noted that with an elaborate mathematical formalism it is possible to quantize non-abelian gauge fields also in the canonical formalism.

8.2 Path integrals

Gaussian integral:

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} dy \exp\left(-\frac{1}{2}ay^2\right) = a^{-1/2}$$

$$(2\pi)^{-n/2} \int_{-\infty}^{\infty} dy_1 \dots dy_n \exp\left(-\frac{1}{2}\vec{y}^T A \vec{y}\right) = (\det A)^{-1/2}$$

(A is a real symmetric positive definite matrix). Since

$$\ln \det A = \text{Tr} \ln A,$$

one has

$$(2\pi)^{-n/2} \int_{-\infty}^{\infty} dy_1 \dots dy_n \exp\left(-\frac{1}{2}\vec{y}^T A \vec{y}\right) = \exp\left(-\frac{1}{2} \text{Tr} \ln A\right)$$

Generalization to an infinite number of components:

$$\int \mathcal{D}\phi \exp\left(-\frac{1}{2} \int dx \int dy \phi(x) A(x,y) \phi(y)\right) = \exp\left(-\frac{1}{2} \text{Tr} \ln A\right)$$

Example:

$$\int \mathcal{D}\eta(t) \exp\left(-\frac{1}{2} \int_{t_i}^{t_f} \eta(t) \left(-\frac{d^2}{dt^2} + \omega^2\right) \eta(t)\right) = \left[\det\left(-\frac{d^2}{dt^2} + \omega^2\right)\right]^{-\frac{1}{2}}$$

To calculate the determinant we solve the eigenvalue problem

$$\left(-\frac{d^2}{dt^2} + \omega^2\right) \Psi_n(t) = \lambda_n \Psi_n(t)$$

One finds

$$\lambda_n = \left(\frac{n\pi}{t_f - t_i}\right)^2 + \omega^2,$$

and therefore

$$\det\left(-\frac{d^2}{dt^2} + \omega^2\right) = \prod_{n=1}^{\infty} \lambda_n = B \frac{\sinh \omega(t_f - t_i)}{\omega(t_f - t_i)}$$

Note: In practice, path integrals are never calculated explicitly !

Gaussian integrals with a linear term:

$$(2\pi)^{-n/2} \int_{-\infty}^{\infty} dy_1 \dots dy_n \exp\left(-\frac{1}{2} \vec{y}^T A \vec{y} + \vec{w}^T \vec{y}\right) = \exp\left(-\frac{1}{2} \text{Tr} \ln A\right) \exp\left(\frac{1}{2} \vec{w}^T A^{-1} \vec{w}\right)$$

(A^{-1} exists because A is positive definite.)

Generalization to an infinite number of components:

$$\int \mathcal{D}\phi \exp\left(-\frac{1}{2} \int dx \int dy \phi(x) A(x, y) \phi(y) + \int dx J(x) \phi(x)\right) = \exp\left(-\frac{1}{2} \text{Tr} \ln A\right) \exp\left(\frac{1}{2} \int dx \int dy J(x) A^{-1}(x, y) J(y)\right)$$

Differentiation with respect to w_i at $\vec{w} = \vec{0}$:

$$\begin{aligned} & \left. \frac{\partial}{\partial w_{i_1}} \dots \frac{\partial}{\partial w_{i_n}} (2\pi)^{-n/2} \int_{-\infty}^{\infty} dy_1 \dots dy_n \exp\left(-\frac{1}{2} \vec{y}^T A \vec{y} + \vec{w}^T \vec{y}\right) \right|_{\vec{w}=0} = \\ & = (2\pi)^{-n/2} \int_{-\infty}^{\infty} dy_1 \dots dy_n y_{i_1} \dots y_{i_n} \exp\left(-\frac{1}{2} \vec{y}^T A \vec{y}\right) \\ & = \left. \frac{\partial}{\partial w_{i_1}} \dots \frac{\partial}{\partial w_{i_n}} \exp\left(-\frac{1}{2} \text{Tr} \ln A\right) \exp\left(\frac{1}{2} \vec{w}^T A^{-1} \vec{w}\right) \right|_{\vec{w}=0} \end{aligned}$$

Functional derivatives:

$$\frac{\delta}{\delta J(y)} Z[J(x)] = \lim_{\varepsilon \rightarrow 0} \frac{Z[J(x) + \varepsilon \delta(x - y)] - Z[J(x)]}{\varepsilon}$$

Generalization to an infinite number of components:

$$\begin{aligned}
& \frac{\partial}{\partial J(x_1)} \cdots \frac{\partial}{\partial J(x_n)} \int \mathcal{D}\phi \exp \left(-\frac{1}{2} \int dx \int dy \phi(x) A(x,y) \phi(y) + \int dx J(x) \phi(x) \right) \Big|_{J=0} = \\
& = \int \mathcal{D}\phi \phi(x_1) \cdots \phi(x_n) \exp \left(-\frac{1}{2} \int dx \int dy \phi(x) A(x,y) \phi(y) \right) \\
& = \frac{\partial}{\partial J(x_1)} \cdots \frac{\partial}{\partial J(x_n)} \exp \left(-\frac{1}{2} \text{Tr} \ln A \right) \exp \left(\frac{1}{2} \int dx \int dy J(x) A^{-1}(x,y) J(y) \right) \Big|_{J=0}
\end{aligned}$$

8.3 Transition amplitudes as path integrals

Let

$$\hat{\phi}_H(t, \vec{x}) = e^{i\hat{H}(t-t_0)} \hat{\phi}_S(t_0, \vec{x}) e^{-i\hat{H}(t-t_0)}$$

Let $|\phi(t_0, \vec{x})\rangle$ be an eigenstate of the Schrödinger field operator $\hat{\phi}_S(t_0, \vec{x})$ with eigenvalue $\phi(t_0, \vec{x})$:

$$\hat{\phi}_S(t_0, \vec{x}) |\phi(t_0, \vec{x})\rangle = \phi(t_0, \vec{x}) |\phi(t_0, \vec{x})\rangle$$

Denote the corresponding Heisenberg state by

$$|\phi(t_0, \vec{x}), t\rangle = e^{i\hat{H}(t-t_0)} |\phi(t_0, \vec{x})\rangle.$$

$|\phi(t_0, \vec{x}), t\rangle$ is an eigenstate of $\hat{\phi}_H$ with eigenvalue $\phi(t_0, \vec{x})$:

$$\hat{\phi}_H(t, \vec{x}) |\phi(t_0, \vec{x}), t\rangle = \phi(t_0, \vec{x}) |\phi(t_0, \vec{x}), t\rangle$$

Proof:

$$\begin{aligned}
\hat{\phi}_H(t, \vec{x}) |\phi(t_0, \vec{x}), t\rangle &= \left(e^{i\hat{H}(t-t_0)} \hat{\phi}_S(t_0, \vec{x}) e^{-i\hat{H}(t-t_0)} \right) \left(e^{i\hat{H}(t-t_0)} |\phi(t_0, \vec{x})\rangle \right) = e^{i\hat{H}(t-t_0)} \hat{\phi}_S(t_0, \vec{x}) |\phi(t_0, \vec{x})\rangle \\
&= e^{i\hat{H}(t-t_0)} \phi(t_0, \vec{x}) |\phi(t_0, \vec{x})\rangle = \phi(t_0, \vec{x}) e^{i\hat{H}(t-t_0)} |\phi(t_0, \vec{x})\rangle = \phi(t_0, \vec{x}) |\phi(t_0, \vec{x}), t\rangle
\end{aligned}$$

As a short-hand notation we write

$$|\phi, t\rangle = |\phi(t_0, \vec{x}), t\rangle.$$

We are interested in the transition amplitude

$$\langle \phi_f, t_f | \phi_i, t_i \rangle$$

$|\phi_i, t_i\rangle$ is a state with eigenvalue $\phi_i(t_0, \vec{x})$ and similar for $|\phi_f, t_f\rangle$. We divide the time interval $(t_f - t_i)$ into $n + 1$ small sub-intervals with time steps at

$$t_i, t_1, t_2, \dots, t_n, t_f.$$

At each intermediate time step we insert a complete set of states

$$\int \mathcal{D}\phi_j(\vec{x}) |\phi_j, t_j\rangle\langle\phi_j, t_j| = 1.$$

Therefore

$$\langle\phi_f, t_f|\phi_i, t_i\rangle = \int \mathcal{D}\phi_n(\vec{x}) \dots \int \mathcal{D}\phi_1(\vec{x}) \langle\phi_f, t_f|\phi_n, t_n\rangle\langle\phi_n, t_n|\phi_{n-1}, t_{n-1}\rangle \dots \langle\phi_1, t_1|\phi_i, t_i\rangle$$

Let us study $\langle\phi_{j+1}, t_{j+1}|\phi_j, t_j\rangle$. If the time interval $(t_{j+1} - t_j)$ is small, we have

$$\begin{aligned} \langle\phi_{j+1}, t_{j+1}|\phi_j, t_j\rangle &= \left(\langle\phi_{j+1}|e^{-i\hat{H}(t_{j+1}-t_j)}\right) \left(e^{i\hat{H}(t_j-t_0)}\phi_j, t_j\right) = \langle\phi_{j+1}|e^{-i\hat{H}(t_{j+1}-t_j)}|\phi_j\rangle \\ &\approx \langle\phi_{j+1}|1 - i(t_{j+1} - t_j)\hat{H}|\phi_j\rangle \end{aligned}$$

Let us first consider a simple case where \hat{H} is replaced by a function $f(\hat{\phi})$, which depends only on $\hat{\phi}$, but not on $\hat{\pi}$. Then

$$\langle\phi_{j+1}|f(\hat{\phi})|\phi_j\rangle = f(\phi_j)\langle\phi_{j+1}|\phi_j\rangle = f(\phi_j)\delta(\phi_{j+1} - \phi_j)$$

We rewrite this as

$$\langle\phi_{j+1}|f(\hat{\phi})|\phi_j\rangle = f(\phi_j) \int \mathcal{D}\pi_j(\vec{x}) \exp\left[i \int d^3x \pi_j(\phi_{j+1} - \phi_j)\right]$$

Note that factors of 2π are absorbed into the integral measure. Next we consider the case where the Hamiltonian is replaced by a function $g(\hat{\pi})$, which depends only on $\hat{\pi}$. We introduce a complete set of momentum eigenstates and obtain

$$\langle\phi_{j+1}|g(\hat{\pi})|\phi_j\rangle = \int \mathcal{D}\pi_j(\vec{x}) g(\pi_j) \exp\left[i \int d^3x \pi_j(\phi_{j+1} - \phi_j)\right]$$

Thus if \hat{H} contains only terms of the form $f(\hat{\phi})$ and $g(\hat{\pi})$, its matrix element can be written as

$$\langle\phi_{j+1}|\hat{H}(\hat{\phi}, \hat{\pi})|\phi_j\rangle = \int \mathcal{D}\pi_j(\vec{x}) H(\phi_j, \pi_j) \exp\left[i \int d^3x \pi_j(\phi_{j+1} - \phi_j)\right]$$

In general this formula will not hold for arbitrary \hat{H} , since the order of a product $\hat{\phi}\hat{\pi}$ matters on the left side (where $\hat{\phi}$ and $\hat{\pi}$ appear as operators), but not on the right side.

If this formula holds, the Hamiltonian is said to be in ‘‘Weyl order’’. Any Hamiltonian can be put into a Weyl order by commuting $\hat{\phi}$ ’s and $\hat{\pi}$ ’s.

We therefore find

$$\begin{aligned} \langle\phi_{j+1}|1 - i(t_{j+1} - t_j)\hat{H}|\phi_j\rangle &= \int \mathcal{D}\pi_j(\vec{x}) (1 - i(t_{j+1} - t_j)H(\phi_j, \pi_j)) \exp\left[i \int d^3x \pi_j(\phi_{j+1} - \phi_j)\right] \\ &= \int \mathcal{D}\pi_j(\vec{x}) \exp\left[i \int d^3x \pi_j(\phi_{j+1} - \phi_j) - i(t_{j+1} - t_j)H(\phi_j, \pi_j)\right] \\ &= \int \mathcal{D}\pi_j(\vec{x}) \exp\left[i(t_{j+1} - t_j)\left(\int d^3x \pi_j \dot{\phi}_j - H(\phi_j, \pi_j)\right)\right] \end{aligned}$$

Therefore

$$\begin{aligned}\langle \phi_f, t_f | \phi_i, t_i \rangle &= \int \mathcal{D}\phi_n(\vec{x}) \dots \int \mathcal{D}\phi_1(\vec{x}) \langle \phi_f, t_f | \phi_n, t_n \rangle \langle \phi_n, t_n | \phi_{n-1}, t_{n-1} \rangle \dots \langle \phi_1, t_1 | \phi_i, t_i \rangle \\ &= \int \mathcal{D}\phi(t, \vec{x}) \int \mathcal{D}\pi(t, \vec{x}) \exp \left[i \int dt \left(\int d^3x \pi \dot{\phi} - H(\phi, \pi) \right) \right]\end{aligned}$$

In most cases $H(\phi, \pi)$ will be quadratic in π . We can then complete the square, perform a Wick rotation and integrate over $\mathcal{D}\pi(t, \vec{x})$. One obtains

$$\begin{aligned}\langle \phi_f, t_f | \phi_i, t_i \rangle &= \int \mathcal{D}\phi(t, \vec{x}) \int \mathcal{D}\pi(t, \vec{x}) \exp \left[i \int dt \left(\int d^3x \pi \dot{\phi} - H(\phi, \pi) \right) \right] \\ &= \int \mathcal{D}\phi(x) \exp \left[i \int d^4x \mathcal{L} \right]\end{aligned}$$

8.4 Correlation function

In the previous section we found

$$\langle \phi_f, t_f | \phi_i, t_i \rangle = \int \mathcal{D}\phi(x) \exp \left[i \int d^4x \mathcal{L} \right]$$

The time interval goes from t_i to t_f , in all other respects this formula is manifestly Lorentz invariant. Any other symmetries that the Lagrangian may have are also explicitly preserved by the functional integral. This will be important for internal symmetries related to gauge groups.

We now would like to make the step and define quantum field theory through path integrals. We have to find a functional formula to compute correlation functions like

$$\langle \Omega | T \hat{\phi}(x_1) \hat{\phi}(x_2) | \Omega \rangle,$$

and to show its equivalence with the canonical operator formalism. Let us consider

$$\int \mathcal{D}\phi(x) \phi(x_1) \phi(x_2) \exp \left[i \int_{-T}^T dt \int d^3x \mathcal{L}(\phi) \right],$$

where the boundary conditions on the path integral are

$$\phi(-T, \vec{x}) = \phi_a(\vec{x}), \quad \phi(T, \vec{x}) = \phi_b(\vec{x}).$$

We break up the functional integral as follows:

$$\int \mathcal{D}\phi(x) = \int \mathcal{D}\phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x}) \int_{\phi(x_1^0, \vec{x}) = \phi_1(\vec{x}), \phi(x_2^0, \vec{x}) = \phi_2(\vec{x})} \mathcal{D}\phi(x)$$

The main functional integral $\int \mathcal{D}\phi(x)$ is now constrained at times x_1^0 and x_2^0 in addition to the endpoints $-T$ and T . With this decomposition the extra factors $\phi(x_1)$ and $\phi(x_2)$ in the original path integral become $\phi_1(\vec{x})$ and $\phi_2(\vec{x})$. If $x_1^0 < x_2^0$:

$$\begin{aligned}
& \int \mathcal{D}\phi(x) \phi(x_1) \phi(x_2) \exp \left[i \int_{-T}^T dt \int d^3x \mathcal{L}(\phi) \right] = \\
& = \int \mathcal{D}\phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x}) \int_{\phi(x_1^0, \vec{x})=\phi_1(\vec{x}), \phi(x_2^0, \vec{x})=\phi_2(\vec{x})} \mathcal{D}\phi(x) \phi(x_1) \phi(x_2) \exp \left[i \int_{-T}^T dt \int d^3x \mathcal{L}(\phi) \right] \\
& = \int \mathcal{D}\phi_1(\vec{x}) \phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x}) \phi_2(\vec{x}) \int_{\phi(x_1^0, \vec{x})=\phi_1(\vec{x}), \phi(x_2^0, \vec{x})=\phi_2(\vec{x})} \mathcal{D}\phi(x) \exp \left[i \int_{-T}^T dt \int d^3x \mathcal{L}(\phi) \right] \\
& = \int \mathcal{D}\phi_1(\vec{x}) \phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x}) \phi_2(\vec{x}) \langle \phi_b, T | \phi_2, x_2^0 \rangle \langle \phi_2, x_2^0 | \phi_1, x_1^0 \rangle \langle \phi_1, x_1^0 | \phi_a, -T \rangle
\end{aligned}$$

Since

$$\hat{\phi}_H(t, \vec{x}) | \phi(\vec{x}), t \rangle = \phi(\vec{x}) | \phi(\vec{x}), t \rangle,$$

we can turn the fields $\phi_i(\vec{x})$ into Heisenberg operators $\hat{\phi}_i(\vec{x})$. Using in addition the completeness relation

$$\int \mathcal{D}\phi(\vec{x}) | \phi, t \rangle \langle \phi, t | = 1,$$

we obtain

$$\begin{aligned}
& \int \mathcal{D}\phi_1(\vec{x}) \phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x}) \phi_2(\vec{x}) \langle \phi_b, T | \phi_2, x_2^0 \rangle \langle \phi_2, x_2^0 | \phi_1, x_1^0 \rangle \langle \phi_1, x_1^0 | \phi_a, -T \rangle = \\
& = \int \mathcal{D}\phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x}) \langle \phi_b, T | \hat{\phi}(x_2) | \phi_2, x_2^0 \rangle \langle \phi_2, x_2^0 | \hat{\phi}(x_1) | \phi_1, x_1^0 \rangle \langle \phi_1, x_1^0 | \phi_a, -T \rangle \\
& = \langle \phi_b, T | \hat{\phi}(x_2) \hat{\phi}(x_1) | \phi_a, -T \rangle
\end{aligned}$$

If we had the order $x_1^0 > x_2^0$ we would have found

$$\langle \phi_b, T | \hat{\phi}(x_1) \hat{\phi}(x_2) | \phi_a, -T \rangle.$$

In summary we have shown that

$$\int_{\phi(-T, \vec{x})=\phi_a(\vec{x}), \phi(T, \vec{x})=\phi_b(\vec{x})} \mathcal{D}\phi(x) \phi(x_1) \phi(x_2) \exp \left[i \int_{-T}^T dt \int d^3x \mathcal{L}(\phi) \right] = \langle \phi_b, T | T \hat{\phi}(x_1) \hat{\phi}(x_2) | \phi_a, -T \rangle$$

If we replace the Heisenberg states by Schrödinger states

$$|\phi(\vec{x}, t)\rangle = e^{i\hat{H}t} |\phi(\vec{x})\rangle,$$

we have

$$\langle \phi_b | e^{-i\hat{H}T} T \hat{\phi}(x_1) \hat{\phi}(x_2) e^{i\hat{H}(-T)} | \phi_a \rangle$$

As in the canonical operator formalism we can now send $T \rightarrow \infty(1 - i\epsilon)$ to project out the ground state

$$e^{-i\hat{H}T} |\phi_a\rangle = \sum_n e^{-iE_n T} |n\rangle \langle n | \phi_a \rangle \rightarrow \langle \Omega | \phi_a \rangle \lim_{T \rightarrow \infty(1-i\epsilon)} e^{-iE_0 T} |\Omega\rangle$$

The phase and the overlap factor drop out if we divide by the same quantity without the field insertions and we obtain the final formula

$$\langle \Omega | T \hat{\phi}(x_1) \hat{\phi}(x_2) | \Omega \rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi(x) \phi(x_1) \phi(x_2) \exp[i \int d^4x \mathcal{L}(\phi)]}{\int \mathcal{D}\phi(x) \exp[i \int d^4x \mathcal{L}(\phi)]}$$

This expresses the two-point correlation function in terms of path integrals. For higher correlation functions one obtains

$$\langle \Omega | T \hat{\phi}(x_1) \dots \hat{\phi}(x_n) | \Omega \rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi(x) \phi(x_1) \dots \phi(x_n) \exp[i \int d^4x \mathcal{L}(\phi)]}{\int \mathcal{D}\phi(x) \exp[i \int d^4x \mathcal{L}(\phi)]}$$

Let us now introduce the “generating functional”

$$Z[J(x)] = \mathcal{N} \int \mathcal{D}\phi(x) \exp \left[i \int d^4x \mathcal{L}(\phi) + J(x) \phi(x) \right]$$

We have

$$\langle \Omega | T \hat{\phi}(x_1) \dots \hat{\phi}(x_n) | \Omega \rangle = \frac{(-i)^n}{Z[0]} \frac{\delta^n Z[J(x)]}{\delta J(x_1) \dots \delta J(x_n)} \Big|_{J=0}$$

The functional $Z[J]$ generates all Green functions:

$$Z[J] = Z[0] \sum_n \frac{i^n}{n!} \int d^4x_1 \dots d^4x_n \langle \Omega | T \hat{\phi}(x_1) \dots \hat{\phi}(x_n) | \Omega \rangle J(x_1) \dots J(x_n)$$

8.5 Fermions in the path integral formalism

8.5.1 Grassmann numbers

Ordinary number commute:

$$[x_i, x_j] = 0.$$

The Grassmann algebra consists of anti-commuting numbers

$$\{\theta_i, \theta_j\} = 0.$$

The differentiation is defined by

$$\frac{\partial}{\partial \theta_j} (\theta_1 \dots \theta_j \dots \theta_m) = (-1)^{j-1} \theta_1 \dots \hat{\theta}_j \dots \theta_m,$$

where the hat indicates that the corresponding variable has to be omitted. Note that

$$\frac{\partial}{\partial \theta_i} \frac{\partial}{\partial \theta_j} F = - \frac{\partial}{\partial \theta_j} \frac{\partial}{\partial \theta_i} F.$$

The Taylor expansion of a function $F(\theta)$ depending on a Grassmann variable θ is given by

$$F(\theta) = F_0 + F_1 \theta.$$

The differential $d\theta$ is also a Grassmann variable:

$$\{\theta, d\theta\} = 0.$$

Integration over a Grassmann variable is defined by

$$\int d\theta = 0, \quad \int d\theta \theta = 1.$$

Multiple integrals are defined by iteration:

$$\int d\theta_1 d\theta_2 F(\theta_1, \theta_2) = \int d\theta_1 \left(\int d\theta_2 F(\theta_1, \theta_2) \right).$$

Let us now consider

$$\begin{aligned} & \int d\theta_1 d\theta_2 \dots d\theta_n d\bar{\theta}_1 d\bar{\theta}_2 \dots d\bar{\theta}_n \exp(\bar{\theta}_i A_{ij} \theta_j) = \\ &= \int d\theta_1 d\theta_2 \dots d\theta_n d\bar{\theta}_1 d\bar{\theta}_2 \dots d\bar{\theta}_n \left[1 + \bar{\theta}_i A_{ij} \theta_j + \dots + \frac{1}{n!} (\bar{\theta}_i A_{ij} \theta_j)^n \right] \\ &= \int d\theta_1 d\theta_2 \dots d\theta_n d\bar{\theta}_1 d\bar{\theta}_2 \dots d\bar{\theta}_n \frac{1}{n!} (\bar{\theta}_i A_{ij} \theta_j)^n \\ &= \int d\theta_1 d\theta_2 \dots d\theta_n d\bar{\theta}_1 d\bar{\theta}_2 \dots d\bar{\theta}_n \frac{1}{n!} (-1)^{\frac{n(n-1)}{2}} \sum_{i_1, \dots, i_n, j_1, \dots, j_n} A_{i_1 j_1} \dots A_{i_n j_n} \bar{\theta}_{i_1} \dots \bar{\theta}_{i_n} \theta_{j_1} \dots \theta_{j_n} \\ &= \int d\theta_1 d\theta_2 \dots d\theta_n d\bar{\theta}_1 d\bar{\theta}_2 \dots d\bar{\theta}_n \frac{1}{n!} (-1)^{\frac{n(n-1)}{2}} \sum_{i_1, \dots, i_n, j_1, \dots, j_n} \varepsilon_{i_1 \dots i_n} \varepsilon_{j_1 \dots j_n} A_{i_1 j_1} \dots A_{i_n j_n} \bar{\theta}_{i_1} \dots \bar{\theta}_{i_n} \theta_{j_1} \dots \theta_{j_n} \\ &= \int d\theta_1 d\theta_2 \dots d\theta_n d\bar{\theta}_1 d\bar{\theta}_2 \dots d\bar{\theta}_n (-1)^{\frac{n(n-1)}{2}} \sum_{j_1, \dots, j_n} \varepsilon_{j_1 \dots j_n} A_{1 j_1} \dots A_{n j_n} \bar{\theta}_1 \dots \bar{\theta}_n \theta_{j_1} \dots \theta_{j_n} \\ &= (-1)^{\frac{n(n-1)}{2}} \sum_{j_1, \dots, j_n} \varepsilon_{j_1 \dots j_n} A_{1 j_1} \dots A_{n j_n} \\ &= (-1)^{\frac{n(n-1)}{2}} \det A. \end{aligned}$$

The limit $n \rightarrow \infty$ yields a path integral over a Grassmann field. We thus arrive at the important formula

$$\det A \sim \int \mathcal{D}\psi(x) \mathcal{D}\bar{\psi}(x) \exp \int d^4x d^4y \bar{\psi}(x) A(x, y) \psi(y).$$

8.5.2 Path integrals with fermions

For fermions we considered up to now the Lagrange density of free fermions and the Lagrange density of interacting fermions in the Yukawa model:

$$\begin{aligned} \mathcal{L}_{Yukawa} &= \mathcal{L}_{Dirac} + \mathcal{L}_{Klein-Gordon} - g\bar{\psi}\psi\phi, \\ \mathcal{L}_{Dirac} &= i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi, \\ \mathcal{L}_{Klein-Gordon} &= \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2. \end{aligned}$$

With the help of Grassmann numbers we may write down the generating functional

$$\begin{aligned} Z[J(x), \eta(x), \bar{\eta}(x)] &= \\ \mathcal{N} \int \mathcal{D}\phi(x) \mathcal{D}\psi(x) \mathcal{D}\bar{\psi}(x) &\exp \left[i \int d^4x \mathcal{L}(\phi) + J(x)\phi(x) + \bar{\psi}(x)\eta(x) + \bar{\eta}(x)\psi(x) \right] \end{aligned}$$

Here $\bar{\psi}(x)$, $\psi(x)$, $\bar{\eta}(x)$ and $\eta(x)$ are fields of Grassmann nature. As before we obtain the Green functions by differentiation. For example:

$$\langle \Omega | T \hat{\psi}_\alpha(x_1) \hat{\psi}_\beta(x_2) | \Omega \rangle = \frac{(-i)^2}{Z[0, 0, 0]} \frac{\delta^2 Z[J(x), \eta(x), \bar{\eta}(x)]}{\delta \bar{\eta}_\alpha(x_1) \delta (-\eta_\beta(x_2))} \Big|_{J=0}$$

The additional minus sign in the differentiation with respect to $\eta(x_2)$ comes from anti-commuting

$$\bar{\psi}(x)\eta(x) = -\eta(x)\bar{\psi}(x).$$

8.6 The reduction formula of Lehmann, Symanzik and Zimmermann

In the canonical operator formalism we stated the formula that the non-trivial part of the S-matrix can be computed as follows

$$\langle \vec{p}_1 \vec{p}_2 \dots | iT | \vec{p}_A \vec{p}_B \rangle = \lim_{T \rightarrow \infty (1-i\epsilon)} \left({}_0 \langle \vec{p}_1 \vec{p}_2 \dots | T \exp \left[-i \int_{-T}^T dt H_I(t) \right] | \vec{p}_A \vec{p}_B \rangle_0 \right)_{connected, amputated}$$

We now derive in the path integral formalism the reduction formula of Lehmann, Symanzik and Zimmermann, which explains why propagators of external legs are amputated. In the path integrals formalism we already showed that

$$\langle \Omega | T \hat{\phi}(x_1) \dots \hat{\phi}(x_n) | \Omega \rangle = \frac{\int \mathcal{D}\phi(x) \phi(x_1) \dots \phi(x_n) \exp [i \int d^4x \mathcal{L}(\phi)]}{\int \mathcal{D}\phi(x) \exp [i \int d^4x \mathcal{L}(\phi)]},$$

where the boundary conditions on the path integral are

$$\lim_{T \rightarrow \infty} \phi(-T, \vec{x}) = \lim_{T \rightarrow \infty} \phi(T, \vec{x}) = 0.$$

For the computation of scattering amplitudes we would like to have as boundary condition not the vacuum but an n particle state. If we assume that interaction are only relevant within a finite volume, we can take this n particle state as the superposition of n non-interacting one-particle states. We call such a state an asymptotic state. Asymptotic fields are solutions of the non-interacting theory, e.g free fields. The general solution for the free field theory is given as a Fourier expansion:

$$\phi_{asympt}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2k_0}} \left(a(k) e^{-ikx} + a^\dagger(k) e^{ikx} \right)$$

With boundary conditions at the remote past/future:

$$\begin{aligned} t \rightarrow +\infty : \quad \phi_{asympt}(x) &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2k_0}} a(k) e^{-ikx}, \\ t \rightarrow -\infty : \quad \phi_{asympt}(x) &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2k_0}} a^\dagger(k) e^{ikx}. \end{aligned}$$

If we give k_0 a small imaginary part $k_0 \rightarrow k_0 - i\epsilon$, we can use the general formula for both cases. Note that $a(k)$ and $a^\dagger(k)$ are here c-numbers, not operators. If we consider a scalar field theory, the asymptotic field satisfies the Klein-Gordon equation

$$(\square + m^2) \phi_{asympt}(x) = 0.$$

Consider now:

$$Z_{asympt}[J] = \int_{\lim \phi = \phi_{asympt}} \mathcal{D}\phi \exp \left[i \int d^4x \mathcal{L}(\phi) + J(x)\phi(x) \right]$$

With

$$\begin{aligned} \mathcal{L}(\phi) &= \mathcal{L}_0(\phi) + \mathcal{L}_{int}(\phi), \\ \exp \left[i \int d^4x \mathcal{L}_{int}(\phi) \right] &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(i \int d^4x \mathcal{L}_{int}(\phi) \right)^n, \\ i\phi(x) \exp \left[i \int d^4y \mathcal{L}_0(\phi) + J(y)\phi(y) \right] &= \frac{\delta}{\delta J(x)} \exp \left[i \int d^4y \mathcal{L}_0(\phi) + J(y)\phi(y) \right], \end{aligned}$$

one arrives at

$$Z_{asympt}[J] = \exp \left[i \int d^4x \mathcal{L}_{int} \left(-i \frac{\delta}{\delta J(x)} \right) \right] \int_{\lim \phi = \phi_{asympt}} \mathcal{D}\phi \exp \left[i \int d^4y \mathcal{L}_0(\phi) + J(y)\phi(y) \right].$$

Let us now define the free-field functional $Z_{asympt,0}[J]$

$$Z_{asympt,0}[J] = \int_{\lim_{\tilde{\phi}=\phi_{asympt}} \tilde{\phi}} \mathcal{D}\tilde{\phi} \exp \left[i \int d^4x \mathcal{L}_0(\tilde{\phi}) + J(x)\tilde{\phi}(x) \right]$$

and change the integration variables according to

$$\tilde{\phi}(x) = \phi(x) + \phi_{asympt}(x).$$

Then

$$Z_{asympt,0}[J] = \int_{\lim_{\phi=0}} \mathcal{D}\phi \exp \left[i \int d^4x \mathcal{L}_0(\phi + \phi_{asympt}) + J(x)\phi(x) + J(x)\phi_{asympt}(x) \right].$$

Note that now the boundary conditions are

$$\lim_{t \rightarrow \pm\infty} \phi = 0.$$

Consider now the example of a scalar field

$$\mathcal{L}_0 = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2.$$

Then

$$\begin{aligned} i \int d^4x \mathcal{L}_0(\phi + \phi_{asympt}) &= i \int d^4x \left[\mathcal{L}_0(\phi) - \frac{1}{2} \phi_{asympt} (\square + m^2) \phi_{asympt} \right. \\ &\quad \left. - \frac{1}{2} \phi (\square + m^2) \phi_{asympt} - \frac{1}{2} [(\square + m^2)] \phi_{asympt} \phi \right] \\ &= i \int d^4x \mathcal{L}_0(\phi) \end{aligned}$$

since ϕ_{asympt} satisfies the Klein-Gordon equation:

$$(\square + m^2) \phi_{asympt} = 0.$$

In this case we have

$$Z_{asympt,0}[J] = \int_{\lim_{\phi=0}} \mathcal{D}\phi \exp \left[i \int d^4y J(y)\phi_{asympt}(y) \right] \exp \left[i \int d^4x \mathcal{L}_0(\phi) + J(x)\phi(x) \right]$$

We now write

$$\int_{\lim_{\phi=0}} \mathcal{D}\phi \exp \left[i \int d^4x \mathcal{L}_0(\phi) + J(x)\phi(x) \right] = \exp \left[-\frac{i}{2} \int d^4y d^4z J(y)\Delta(y,z)J(z) \right]$$

For the example discussed above (scalar field theory) we have

$$\Delta(x-y) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip(x-y)} \frac{1}{p^2 - m^2 + i\epsilon}$$

Since

$$(\square_x + m^2) \Delta(x-y) = -\delta(x-y)$$

and

$$\begin{aligned} \frac{\delta}{\delta J(x)} \exp \left[-\frac{i}{2} \int d^4 y d^4 z J(y) \Delta(y,z) J(z) \right] = \\ -i \int d^4 w \Delta(x-w) J(w) \exp \left[-\frac{i}{2} \int d^4 y d^4 z J(y) \Delta(y,z) J(z) \right] \end{aligned}$$

we have

$$(\square_x + m^2) \frac{\delta}{\delta J(x)} \exp \left[-\frac{i}{2} \int d^4 y d^4 z J(y) \Delta(y,z) J(z) \right] = iJ(x) \exp \left[-\frac{i}{2} \int d^4 y d^4 z J(y) \Delta(y,z) J(z) \right]$$

Therefore

$$\begin{aligned} Z_{asympt}[J] &= \exp \left[\int d^4 x \phi_{asympt}(x) \cdot (\square_x + m^2) \frac{\delta}{\delta J(x)} \right] \\ &\times \exp \left[i \int d^4 y \mathcal{L}_{int} \left(-i \frac{\delta}{\delta J(y)} \right) \right] \\ &\times \int_{\lim \phi=0} \mathcal{D}\phi \exp \left[i \int d^4 z (\mathcal{L}_0(\phi) + J(z)\phi(z)) \right]. \end{aligned}$$

Let us now define

$$Z[J] = \int_{\lim \phi=0} \mathcal{D}\phi \exp \left[i \int d^4 z \mathcal{L}(\phi) + J(z)\phi(z) \right].$$

Define the Green functions as functional derivatives of $Z[J]$:

$$\begin{aligned} G^n(x_1, \dots, x_n) &= \langle 0 | T \hat{\phi}(x_1) \dots \hat{\phi}(x_n) | 0 \rangle \\ &= \frac{(-i)^n}{Z[0]} \frac{\delta^n Z[J]}{\delta J(x_1) \dots \delta J(x_n)} \Big|_{J=0} \end{aligned}$$

Then

$$Z_{asympt}[0] = \sum \frac{i^n}{n!} \int d^4 x_1 \dots d^4 x_n \phi_{asympt}(x_1) \dots \phi_{asympt}(x_n) (\square_{x_1} + m^2) \dots (\square_{x_n} + m^2) G^n(x_1, \dots, x_n).$$

Define now the Fourier transform of the Green functions by

$$G^n(x_1, \dots, x_n) = \int \frac{d^4 p_1}{(2\pi)^4} \dots \frac{d^4 p_n}{(2\pi)^4} e^{-i \sum p_j x_j} (2\pi)^4 \delta^4(p_1 + \dots + p_n) \tilde{G}^n(p_1, \dots, p_n)$$

and the truncated (amputated) Green function in momentum space by

$$\tilde{G}_{trunc}^n(p_1, \dots, p_n) = \left(\frac{i}{p_1^2 - m^2} \right)^{-1} \dots \left(\frac{i}{p_n^2 - m^2} \right)^{-1} \tilde{G}^n(p_1, \dots, p_n).$$

Then

$$Z_{asympt}[0] = \sum \frac{1}{n!} \int \frac{d^4 p_j}{(2\pi)^4} (2\pi)^4 \delta^4(\sum p_k) \tilde{G}_{trunc}^n(p_1, \dots, p_n) \int d^4 x_1 \dots d^4 x_n e^{-i \sum p_j x_j} \phi_{asympt}(x_1) \dots \phi_{asympt}(x_n).$$

Consider

$$\begin{aligned} \int d^4 x e^{-i p x} \phi_{asympt}(x) &= \int d^4 x e^{-i p x} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2k_0}} \left[a(k) e^{-i k x} + a^\dagger(k) e^{i k x} \right] \\ &= \int d^3 k \frac{2\pi}{\sqrt{2k_0}} \left[a(k) \delta^4(p+k) + a^\dagger(k) \delta^4(p-k) \right] \end{aligned}$$

For $k_0 > 0$ and $p_0 > 0$ only the second term contributes. For $k_0 > 0$ and $p_0 < 0$ only the first term contributes. One obtains

$$Z_{asympt}[0] = \sum \frac{1}{n!} \int \frac{d^3 p_j}{(2\pi)^3} \frac{1}{\sqrt{|2p_0|}} (2\pi)^4 \delta^4(\sum p_k) \tilde{G}_{trunc}^n(p_1, \dots, p_n) \prod a(-p_i) \prod a^\dagger(p_j)$$

9 Gauge theories

We have seen that electrodynamics can be described by a gauge potential $A_\mu(x)$.

$$\mathcal{L}(A_\mu, \partial_\mu A_\nu) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

This Lagrange density is invariant under the gauge transformation

$$A_\mu(x) \rightarrow A_\mu(x) - \partial_\mu \Lambda(x).$$

We can write this gauge transformation also as

$$A_\mu(x) \rightarrow U(x) (A_\mu(x) + i\partial_\mu) U^\dagger(x),$$

with

$$U(x) = e^{-i\Lambda(x)}.$$

The gauge symmetry is given by a $U(1)$ group: This is an abelian group, whose elements can be parameterized with a coordinate φ as follows:

$$e^{-i\varphi}, \quad 0 \leq \varphi < 2\pi.$$

This is obviously a group:

$$\begin{aligned} e^{-i\varphi_1} \cdot e^{-i\varphi_2} &= e^{-i(\varphi_1 + \varphi_2)}, \\ (e^{-i\varphi})^{-1} &= e^{i\varphi}. \end{aligned}$$

It is also a one-dimensional compact manifold (e.g. the circle line).

9.1 Lie groups und Lie algebras

A Lie group is a group G which is also an analytic manifold such that the mapping $(a, b) \rightarrow ab^{-1}$ of the product manifold $G \times G$ into G is analytic.

A Lie algebra over a commutative ring K is a K -module A together with a mapping $x \otimes y \rightarrow [x, y]$ such that for $x, y, z \in A$:

$$\begin{aligned} [x, x] &= 0 \\ [x, [y, z]] + [y, [z, x]] + [z, [x, y]] &= 0 \end{aligned}$$

Elements of a Lie group are written in terms of the generators as

$$g = \exp(-iT^a \alpha_a)$$

The generators T^a satisfy a Lie algebra, e.g. the commutators of generators are linear combinations of the generators, i.e.

$$[T^a, T^b] = if^{abc}T^c$$

Note that in the mathematical literature the convention for the definition of the generators is usually such that no explicit factors of i appear in the formulae above. The convention used in the physical literature (which is adopted here) ensures that the generators for the unitary groups are hermitian matrices.

Examples of Lie groups:

- $GL(n, \mathbb{R}), GL(n, \mathbb{C})$: group of non-singular $n \times n$ matrices with n^2 real parameters ($GL(n, \mathbb{R})$), respectively $2n^2$ real parameters ($GL(n, \mathbb{C})$).
- $SL(n, \mathbb{R}), SL(n, \mathbb{C})$: $\det A = 1$; $SL(n, \mathbb{R})$ has $n^2 - 1$ real parameters; $SL(n, \mathbb{C})$ has $2(n^2 - 1)$ real parameters.
- $O(n)$: $RR^T = 1$
- $SO(n)$: $RR^T = 1$ and $\det R = 1$.
- $U(n)$: $UU^\dagger = 1$; n^2 real parameters.
- $SU(n)$: $UU^\dagger = 1$ and $\det U = 1$; $n^2 - 1$ real parameters.
- $Sp(n)$: Invariance group of

$$\sum_{j=1}^n (x_j y_{j+n} - x_{j+n} y_j)$$

A Lie algebra is simple if it is non-Abelian and has no non-trivial ideals.

A Lie algebra is called semi-simple if it has no non-trivial Abelian ideals.

A simple Lie algebra is also semi-simple.

Semi-simple groups are a direct product of simple groups. The compact simple Lie algebras are

$$\begin{aligned} A_n &= SU(n+1), \\ B_n &= SO(2n+1), \\ C_n &= Sp(n), \\ D_n &= SO(2n). \end{aligned}$$

The exceptional groups are

$$E_6, E_7, E_8, F_4, G_2.$$

9.2 Special unitary Lie groups

We discuss here $SU(N)$. The standard normalization is

$$\text{Tr}(T^a T^b) = T_R \delta^{ab} = \frac{1}{2} \delta^{ab}$$

As an example we consider first the group $SU(2)$. This group has three generators I^1, I^2 und I^3 , which are proportional to the Pauli matrices:

$$I^1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad I^2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad I^3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

As a further example we consider $SU(3)$. Here we have eight generators $\lambda^a, a = 1, \dots, 8$, which are called Gell-Mann matrices.

$$\begin{aligned} \lambda^1 &= \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda^2 &= \frac{1}{2} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda^3 &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \lambda^4 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \lambda^5 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & \lambda^6 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ \lambda^7 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \lambda^8 &= \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \end{aligned}$$

The Fierz identity reads for $SU(N)$:

$$T_{ij}^a T_{kl}^a = \frac{1}{2} \left(\delta_{il} \delta_{jk} - \frac{1}{N} \delta_{ij} \delta_{kl} \right).$$

Proof: T^a and the unit matrix form a basis of the $N \times N$ hermitian matrices, therefore any hermitian matrix A can be written as

$$A = c_0 1 + c_a T^a.$$

The constants c_0 and c_a are determined using the normalization condition and the fact that the T^a are traceless:

$$\begin{aligned} c_0 &= \frac{1}{N} \text{Tr}(A), \\ c_a &= 2 \text{Tr}(T^a A). \end{aligned}$$

Therefore

$$A_{lk} \left(2T_{ij}^a T_{kl}^a + \frac{1}{N} \delta_{ij} \delta_{kl} - \delta_{il} \delta_{jk} \right) = 0.$$

This has to hold for an arbitrary A , therefore the Fierz identity follows. Useful formulae involving traces:

$$\begin{aligned}\mathrm{Tr}(T^a X) \mathrm{Tr}(T^a Y) &= \frac{1}{2} \left[\mathrm{Tr}(XY) - \frac{1}{N} \mathrm{Tr}(X) \mathrm{Tr}(Y) \right], \\ \mathrm{Tr}(T^a X T^a Y) &= \frac{1}{2} \left[\mathrm{Tr}(X) \mathrm{Tr}(Y) - \frac{1}{N} \mathrm{Tr}(XY) \right].\end{aligned}$$

From

$$[T^a, T^b] = if^{abc} T^c$$

one derives by multiplying with T^d and taking the trace:

$$if^{abc} = 2 \left[\mathrm{Tr}(T^a T^b T^c) - \mathrm{Tr}(T^b T^a T^c) \right]$$

This yields an expression of the structure constants in terms of the matrices of the fundamental representation. We can now calculate for the group $SU(N)$ the fundamental and the adjoint Casimirs:

$$\begin{aligned}(T^a T^a)_{ij} &= C_F \delta_{ij} = \frac{N^2 - 1}{2N} \delta_{ij}, \\ f^{abc} f^{dbc} &= C_A \delta^{ad} = N \delta^{ad}.\end{aligned}$$

9.3 Yang-Mills theory

C.N. Yang and R.L.Mills¹ suggested 1954 a generalization towards non-abelian gauge groups. The field strength tensor is now given by

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c,$$

where a is an index running from 1 to $N^2 - 1$ for a $SU(N)$ gauge group. The Lagrange density reads:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}.$$

The Lagrange density is invariant under the local transformations

$$T^a A_\mu^a(x) \rightarrow U(x) \left(T^a A_\mu^a(x) + \frac{i}{g} \partial_\mu \right) U^\dagger(x)$$

with

$$U(x) = \exp(-iT^a \theta_a(x)).$$

The action is given by the integral over the Lagrange density:

$$S = \int d^4x \mathcal{L}$$

¹C.N. Yang and R.L. Mills, Phys. Rev. 96, (1954), 191

9.4 Quantization of gauge theories

Let us examine closer the quantization of gauge theories. The Lagrange density is given by

$$\mathcal{L}_{QCD} = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu},$$

with

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c.$$

This Lagrange density is nicely invariant under local gauge transformations

$$T^a A_\mu^a(x) \rightarrow U(x) \left(T^a A_\mu^a(x) + \frac{i}{g} \partial_\mu \right) U^\dagger(x),$$

but we are not yet happy: If we try to calculate the gluon propagator we have to invert a certain matrix and it turns out that this matrix is singular. Something is going wrong. Let's look again at the generating functional:

$$Z[J] = \int \mathcal{D}A_\mu^a(x) \exp \left[i \int d^4x \mathcal{L} + A_\mu^a(x) J_\mu^a(x) \right]$$

The path integral is over all possible gauge field configurations, even the ones which are just related by a gauge transformation. These configuration describe the same physics and it is sufficient to count them only ones. Technically this is done as follows: Let us denote a gauge transformation by

$$U(x) = \exp \left(-iT^b \theta_b(x) \right).$$

The gauge transformation is therefore completely specified by the functions $\theta_b(x)$. We denote by $A_\mu^a(x, \theta_b)$ the gauge field configuration obtained from $A_\mu^a(x)$ through the gauge transformation $U(x)$:

$$T^a A_\mu^a(x, \theta_b) = U(x) \left(T^a A_\mu^a(x) + \frac{i}{g} \partial_\mu \right) U^\dagger(x),$$

$A_\mu^a(x, \theta_b)$ and $A_\mu^a(x)$ are therefore gauge-equivalent configurations. From all gauge-equivalent configurations we are going to pick the one, which satisfies for a given G^μ and $B^a(x)$ the equation

$$G^\mu A_\mu^a(x, \theta_b) = B^a(x).$$

Let us assume that this equation gives a unique solution θ_b for a given A_μ^a . (This is not necessarily always fulfilled, cases where a unique solution may not exist are known as the Gribov ambiguity.) Let α_j , $j = 1, \dots, n$ be a n -dimensional discrete vector and let the

$$g_i = g_i(\alpha_1, \dots, \alpha_n), \quad i = 1, \dots, n,$$

be functions of α_j . Then

$$\int \left(\prod_{j=1}^n d\alpha_j \right) \left(\prod_{i=1}^n \delta(g_i(\alpha_1, \dots, \alpha_n)) \right) \det \left(\frac{\partial g_i}{\partial \alpha_j} \right) = 1.$$

Proof: We change the variables from α_j to

$$\beta_i = g_i(\alpha_1, \dots, \alpha_n).$$

Then

$$\int \left(\prod_{j=1}^n d\alpha_j \right) \left(\prod_{i=1}^n \delta(g_i(\alpha_1, \dots, \alpha_n)) \right) \det \left(\frac{\partial g_i}{\partial \alpha_j} \right) = \int \left(\prod_{j=1}^n d\beta_j \right) \left(\prod_{i=1}^n \delta(\beta_i) \right) = 1.$$

We generalize this to the continuum. For a gauge theory with a single generator we obtain:

$$\int \mathcal{D}\theta(x) \delta(G^\mu A_\mu(x, \theta(x)) - B(x)) \det \left(\frac{\delta G^\mu A_\mu(x, \theta(x))}{\delta \theta(y)} \right) = 1.$$

For a gauge theory with n generators we find

$$\int \prod_b \mathcal{D}\theta_b(x) \delta^n(G^\mu A_\mu^a(x, \theta_b(x)) - B^a(x)) \det M_G = 1$$

where

$$(M_G(x, y))_{ab} = \frac{\delta G^\mu A_\mu^a(x, \theta_c(x))}{\delta \theta_b(y)}$$

Remark: θ_b are coordinates of the Lie group:

$$U = \exp(-iT^b \theta_b)$$

As the Lie group is also a manifold, we can integrate over the manifold. With the coordinates above, the invariant measure is given by

$$\prod_b \mathcal{D}\theta_b.$$

The integral measure dg is called a left invariant measure, if

$$\int dg f(g_0 g) = \int dg f(g)$$

for arbitrary elements g and g_0 of the group G . A measure is called right invariant, if

$$\int dg f(g g_0) = \int dg f(g)$$

In general, right and left invariant measures are not necessarily equal. However, it is known that they are equal for compact groups, simple groups and semi-simple groups.

Remark 2: If the gauge fixing condition is chosen such that

$$G^\mu A_\mu^a(x, \theta_b) - B^a(x) = 0$$

is linear in θ , then the functional derivative

$$\frac{\delta G^\mu A_\mu^a(x, \theta_c(x))}{\delta \theta_b(y)}$$

will be independent of θ and we may take the determinant in front of the integral

$$\det M_G \int \prod_b \mathcal{D}\theta_b(x) \delta^n (G^\mu A_\mu^a(x, \theta_b(x)) - B^a(x)) = 1.$$

We now consider

$$Z[0] = \int \mathcal{D}A_\mu^a(x) \exp \left[i \int d^4x \mathcal{L}(A_\mu^a(x)) \right]$$

and insert the above equation

$$\begin{aligned} Z[0] &= \\ &= \int \mathcal{D}A_\mu^a(x) \det M_G(A_\mu^a(x)) \int \prod_b \mathcal{D}\theta_b(x) \delta^n (G^\mu A_\mu^a(x, \theta_b(x)) - B^a(x)) \exp \left[i \int d^4x \mathcal{L}(A_\mu^a(x)) \right] \\ &= \int \prod_b \mathcal{D}\theta_b(x) \int \mathcal{D}A_\mu^a(x) \det M_G(A_\mu^a(x, \theta_b(x))) \delta^n (G^\mu A_\mu^a(x, \theta_b(x)) - B^a(x)) \\ &\quad \times \exp \left[i \int d^4x \mathcal{L}(A_\mu^a(x, \theta_b(x))) \right] \\ &= \int \prod_b \mathcal{D}\theta_b(x) \int \mathcal{D}A_\mu^a(x, \theta_b(x)) \det M_G(A_\mu^a(x, \theta_b(x))) \delta^n (G^\mu A_\mu^a(x, \theta_b(x)) - B^a(x)) \\ &\quad \times \exp \left[i \int d^4x \mathcal{L}(A_\mu^a(x, \theta_b(x))) \right] \\ &= \left(\int \prod_b \mathcal{D}\theta_b(x) \right) \int \mathcal{D}A_\mu^a(x) \det M_G(A_\mu^a(x)) \delta^n (G^\mu A_\mu^a(x) - B^a(x)) \exp \left[i \int d^4x \mathcal{L}(A_\mu^a(x)) \right] \end{aligned}$$

Here we used the gauge invariance of the action, of $\det M_G$ and of $\mathcal{D}A_\mu^a(x)$. The integral over all gauge-transformations

$$\int \prod_b \mathcal{D}\theta_b(x)$$

is the just an irrelevant prefactor, which we neglect in the following. We then obtain

$$Z[0] = \int \mathcal{D}A_\mu^a(x) \det M_G(A_\mu^a(x)) \delta^n(G^\mu A_\mu^a(x) - B^a(x)) \exp \left[i \int d^4x \mathcal{L}(A_\mu^a(x)) \right]$$

This functional still depends on $B^a(x)$. As we are not interested in any particular choice of $B^a(x)$, we average over $B^a(x)$ with weight

$$\exp \left(-\frac{i}{2\xi} \int d^4x (B^a(x) B_a(x)) \right)$$

and obtain

$$\begin{aligned} & \int \mathcal{D}B^a(x) Z[0] \exp \left(-\frac{i}{2\xi} \int d^4x (B^a(x) B_a(x)) \right) = \\ & = \int \mathcal{D}B^a(x) \int \mathcal{D}A_\mu^a(x) \det M_G \delta^n(G^\mu A_\mu^a(x) - B^a(x)) \exp \left[i \int d^4x \mathcal{L}(A_\mu^a(x)) - \frac{1}{2\xi} B^a(x) B_a(x) \right] \\ & = \int \mathcal{D}A_\mu^a(x) \det M_G \exp \left[i \int d^4x \mathcal{L}(A_\mu^a(x)) - \frac{1}{2\xi} (G^\mu A_\mu^a(x)) (G^\nu A_{\nu a}(x)) \right] \end{aligned}$$

We now consider as new generating functional

$$Z[J] = \int \mathcal{D}A_\mu^a(x) \det M_G \exp \left[i \int d^4x \mathcal{L}(A_\mu^a(x)) - \frac{1}{2\xi} (G^\mu A_\mu^a(x)) (G^\nu A_{\nu a}(x)) + A_\mu^a(x) J_a^\mu(x) \right]$$

We observe that $Z[J]$ contains a term we could expect from a naive fixing of the gauge

$$-\frac{1}{2\xi} (G^\mu A_\mu^a(x)) (G^\nu A_{\nu a}(x)).$$

In addition, the determinant

$$\det M_G$$

appears in front of the exponent.

Various gauges are:

- Lorentz gauge or covariant gauge: $G^\mu = \partial^\mu$.

$$(M_G(x,y))^{ab} = \left(\delta^{ab} \square - g f^{abc} \partial^\mu A_\mu^c \right) \delta^4(x-y)$$

- Coulomb gauge: $G^\mu = (0, \vec{\nabla})$.

$$(M_G(x,y))^{ab} = \left(\delta^{ab} \nabla^2 - g f^{abc} \vec{A}^c \vec{\nabla} \right) \delta^4(x-y)$$

- Axial gauge: $G^\mu = n^\mu$, where n^μ is a constant four-vector.

$$(M_G(x,y))^{ab} = \left(\delta^{ab} n \cdot \partial - g f^{abc} n \cdot A^c \right) \delta^4(x-y)$$

We would like to exponentiate the determinant. In the treatment of fermions within the path-integral formalism we had the formula

$$\det A \sim \int \mathcal{D}\Psi(x) \mathcal{D}\bar{\Psi}(x) \exp \int d^4x d^4y \bar{\Psi}(x) A(x,y) \Psi(y).$$

We write this as

$$\det A = \int \mathcal{D}\Psi(x) \mathcal{D}\bar{\Psi}(x) \exp -i \int d^4x d^4y \bar{\Psi}(x) A(x,y) \Psi(y).$$

Applying this to $\det M_G$:

$$\det M_G = \int \mathcal{D}c^b(x) \mathcal{D}\bar{c}^a(x) \exp \left(i \int d^4x \bar{c}^a(x) \left(-M_G^{ab} \right) c^b(x) \right).$$

Specializing to the covariant gauge $G^\mu = \partial^\mu$ one obtains

$$\begin{aligned} \det M_G &= \int \mathcal{D}c^b(x) \mathcal{D}\bar{c}^a(x) \exp \left(i \int d^4x \bar{c}^a(x) \left(-\delta^{ab} \square + g f^{abc} \partial^\mu A_\mu^c \right) c^b(x) \right) \\ &= \int \mathcal{D}c^b(x) \mathcal{D}\bar{c}^a(x) \exp \left(i \int d^4x \bar{c}^a(x) \left(-\partial^\mu D_\mu^{ab} \right) c^b(x) \right), \end{aligned}$$

where

$$D_\mu^{ab} = \delta^{ab} \partial_\mu - g f^{abc} A_\mu^c$$

is the covariant derivative. In the Lorenz gauge we obtain finally

$$\begin{aligned} Z[J, \xi, \bar{\xi}] &= \int \mathcal{D}A_\mu^a(x) \int \mathcal{D}c^b(x) \mathcal{D}\bar{c}^a(x) \\ &\exp \left[i \int d^4x \mathcal{L}(A_\mu^a(x)) - \frac{1}{2\xi} (\partial^\mu A_\mu^a(x)) (\partial^\nu A_{\nu a}(x)) + \bar{c}^a(x) \left(-\partial^\mu D_\mu^{ab} \right) c^b(x) \right. \\ &\left. + A_\mu^a(x) J_a^\mu(x) + \bar{c}^a(x) \xi_a(x) + \bar{\xi}_a(x) c^a(x) \right] \end{aligned}$$

9.5 The Lagrange density for the fermion sector

Example: The Lagrange density for a free electron (e.g. no interactions) is given by

$$\mathcal{L}_F = \bar{\Psi} (i \gamma^\mu \partial_\mu - m_e) \Psi.$$

We are now looking for a Lagrange density for the fermionic sector, which remains invariant under gauge transformations. Recall that under a gauge transformation a fermion field $\psi_i(x)$ transforms as

$$\begin{aligned}\psi_i(x) &\rightarrow U_{ij}(x)\psi_j(x) & U_{ij}(x) &= \exp(-iT^a\theta^a(x)). \\ \bar{\psi}_i(x) &\rightarrow \bar{\psi}_j(x)U_{ji}^\dagger(x)\end{aligned}$$

$\theta^a(x)$ depends on the space-time coordinates x . For an infinitesimal gauge transformation we have

$$\psi_i(x) \rightarrow (1 - iT^a\theta^a(x))\psi_j(x).$$

We immediately see that a fermion mass term

$$-m\bar{\psi}(x)\psi(x)$$

is invariant under gauge transformations. (Note however that in the standard model the fermion masses are generated through the Yukawa couplings to the Higgs field.) But the term involving derivatives is not gauge invariant:

$$i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) \rightarrow i\bar{\psi}(x)U^\dagger(x)\gamma^\mu\partial_\mu(U(x)\psi(x)) = i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) + \underbrace{i\bar{\psi}(x)\gamma^\mu\left(U^\dagger(x)\partial_\mu U(x)\right)}_{extra}\psi(x).$$

The solution comes in the form of the covariant derivative

$$D_\mu = \partial_\mu - igT^aA_\mu^a(x),$$

where the gauge field transforms as

$$T^aA_\mu^a(x) \rightarrow U(x)\left(T^aA_\mu^a(x) + \frac{i}{g}\partial_\mu\right)U^\dagger(x)$$

Then the combination

$$\begin{aligned}i\bar{\psi}(x)\gamma^\mu D_\mu\psi(x) &= i\bar{\psi}(x)\gamma^\mu(\partial_\mu - igT^aA_\mu^a(x))\psi(x) \\ &\rightarrow i\bar{\psi}(x)U^\dagger(x)\gamma^\mu\left[\partial_\mu - igU(x)\left(T^aA_\mu^a(x) + \frac{i}{g}\partial_\mu\right)U^\dagger(x)\right]U(x)\psi(x) \\ &= i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) + i\bar{\psi}(x)\gamma^\mu\left(U^\dagger(x)\partial_\mu U(x)\right)\psi(x) \\ &\quad + g\bar{\psi}(x)\gamma^\mu T^aA_\mu^a(x)\psi(x) + i\bar{\psi}(x)\gamma^\mu\left[\left(\partial_\mu U^\dagger(x)\right)U(x)\right]\psi(x)\end{aligned}$$

is invariant.

The Lagrange density for the fermion sector:

$$\mathcal{L}_{fermions} = \sum_{fermions} \bar{\psi}(x)(i\gamma^\mu D_\mu - m)\psi(x).$$

9.6 Feynman rules for QED and QCD

Expanding the Lagrange density into terms bilinear in the fields and interaction terms. As an example consider the gluonic part of the Lagrange density:

$$\mathcal{L}_{QCD} = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2\xi}(\partial^\mu A_\mu^a)^2 + \mathcal{L}_{FP} + \mathcal{L}_{fermions}$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c.$$

This yields

$$\begin{aligned} \mathcal{L}_{QCD} = & -\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 - \frac{1}{2\xi}(\partial^\mu A_\mu^a)^2 \\ & - g f^{abc} (\partial_\mu A_\nu^a) A^{b\mu} A^{c\nu} - \frac{1}{4}g^2 (f^{eab} A_\mu^a A_\nu^b) (f^{ecd} A^{c\mu} A^{d\nu}) \\ & + \mathcal{L}_{FP} + \mathcal{L}_{fermions}. \end{aligned}$$

Terms bilinear in the fields define the propagators. Terms with three or more fields define interaction vertices.

For QED (photons and electrons) the Lagrange density is given by

$$\mathcal{L}_{QED} = \bar{\Psi}(i\cancel{\partial} - m_e)\Psi - \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 + e\bar{\Psi}\gamma^\mu A_\mu\Psi$$

9.6.1 Propagators

Terms bilinear in the fields yield propagators. Consider

$$\begin{aligned} \mathcal{L}_{bilinear}(x) = & \\ & \frac{1}{2}\phi_i(x)P_{ij}^{boson,real}(x)\phi_j(x) + \chi_i^*(x)P_{ij}^{boson,complex}(x)\chi_j(x) + \bar{\Psi}_i(x)P_{ij}^{fermion}(x)\Psi_j(x). \end{aligned}$$

where ϕ_i denotes a set of real boson fields (one degree of freedom), χ_i denotes a set of complex boson fields (two degrees of freedom), and ψ_i denotes a set of fermion fields. The boson fields may be scalar or vector fields. P is a matrix operator that may contain derivatives and must have an inverse. P is taken to be a hermitian operator (real symmetric operator in the case of real boson fields).

$$P^\dagger = P$$

Define the inverse of P by

$$\sum_j P_{ij}(x)P_{jl}^{-1}(x-y) = \delta_{il}\delta^4(x-y)$$

and its Fourier transform

$$P_{ij}^{-1}(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot x} \tilde{P}_{ij}^{-1}(k)$$

Then the propagator is given by

$$\Delta_F(k)_{ij} = \int d^4x e^{ik \cdot x} \langle 0 | T(\phi_i(x) \phi_j(0)) | 0 \rangle = i(\tilde{P}^{-1}(k))_{ij}$$

The propagator of a scalar particle:

From

$$P(x) = -\square - m^2$$

we found already

$$\tilde{P}(k) = \frac{1}{k^2 - m^2}.$$

and therefore

$$\Delta_F(k) = \frac{i}{k^2 - m^2}.$$

The propagator of a fermion:

$$P(x) = i\partial\!\!\!/ - m$$

Then

$$(i\partial\!\!\!/ - m) \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot (x-y)} \tilde{P}^{-1}(k) = \int \frac{d^4k}{(2\pi)^4} (\not{k} - m) e^{-ik \cdot (x-y)} \tilde{P}^{-1}(k) = \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot (x-y)}.$$

Therefore

$$(\not{k} - m) \tilde{P}^{-1}(k) = 1$$

and

$$\Delta_F(k) = i \frac{\not{k} + m}{k^2 - m^2}$$

The propagator of a gauge boson:

In Lorentz gauge the gauge-fixing term is given by

$$\mathcal{L}_{GF} = -\frac{1}{2\xi} (\partial^\mu A_\mu)^2$$

and therefore

$$P(x) = \square g^{\mu\nu} - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu$$

Then

$$P(x) \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot (x-y)} \tilde{P}^{-1}(k) = \int \frac{d^4 k}{(2\pi)^4} k^2 \left(-g^{\mu\nu} + \left(1 - \frac{1}{\xi}\right) \frac{k^\mu k^\nu}{k^2} \right) e^{-ik \cdot (x-y)} \tilde{P}^{-1}(k).$$

We have for

$$M_{\mu\nu} = -g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right) \frac{k_\mu k_\nu}{k^2} \quad \text{and} \quad N_{\mu\nu} = -g_{\mu\nu} + (1 - \xi) \frac{k_\mu k_\nu}{k^2}$$

the following relation:

$$M_{\mu\lambda} N^{\lambda\nu} = g_\mu^\nu.$$

Therefore

$$\Delta_F(k) = \frac{i}{k^2} \left(-g_{\mu\nu} + (1 - \xi) \frac{k_\mu k_\nu}{k^2} \right)$$

9.6.2 Vertices

A general term in $\mathcal{L}_{int}(x)$ has the form

$$\mathcal{L}_{int}(x) = \int d^4 x_1 \dots d^4 x_n \alpha_{i_1 \dots i_n}(x, x_1, \dots, x_n) \phi_{i_1}(x_1) \dots \phi_{i_n}(x_n)$$

For the vertex we define

$$\alpha_{i_1 \dots i_n}(x, x_1, \dots, x_n) = \int \frac{d^4 k_1}{(2\pi)^4} \dots \frac{d^4 k_n}{(2\pi)^4} e^{-ik_1(x-x_1) - \dots - ik_n(x-x_n)} \tilde{\alpha}_{i_1 \dots i_n}(k_1, \dots, k_n)$$

$\tilde{\alpha}$ contains a factor ik_{j_μ} for every derivative $\partial/\partial x_{j_\mu}$ acting on a field with argument x_j . The vertex is then given by

$$I(k_1, \dots, k_n) = i \sum_{\text{permutations}} (-1)^P \tilde{\alpha}_{i_1 \dots i_n}(k_1, \dots, k_n)$$

The summations are over all permutations of indices and momenta. The momenta are taken to flow inward.

Example: The quark-gluon vertex

$$\begin{aligned} \mathcal{L}_{int} &= g \bar{\Psi}^i \gamma^\mu (T^a)^{ij} A_\mu^a \Psi^j = \int d^4 x_1 \int d^4 x_2 \int d^4 x_3 \delta^4(x-x_1) \delta^4(x-x_2) \delta^4(x-x_3) \\ &\quad \times [g \gamma^\mu (T^a)^{ij}] \bar{\Psi}^i(x_1) \Psi^j(x_2) A_\mu^a(x_3) \end{aligned}$$

Then

$$\tilde{\alpha}(k_1, k_2, k_3) = g\gamma^\mu(T^a)^{ij}$$

and

$$I = ig\gamma^\mu(T^a)^{ij}$$

Example 2: The three-gluon vertex

$$\begin{aligned}\mathcal{L}_{int} &= -gf^{abc} (\partial_\mu A_\nu^a) A^{b\mu} A^{c\nu} \\ &= \int d^4x_1 \int d^4x_2 \int d^4x_3 \delta^4(x-x_1) \delta^4(x-x_2) \delta^4(x-x_3) \left(-gf^{abc} \right) \partial_{x_1}^\nu g^{\mu\lambda} A_\mu^a(x_1) A_\nu^b(x_2) A_\lambda^c(x_3) \\ &= \int d^4x_1 \int d^4x_2 \int d^4x_3 A_\mu^a(x_1) A_\nu^b(x_2) A_\lambda^c(x_3) gf^{abc} g^{\mu\lambda} \partial_{x_1}^\nu \delta^4(x-x_1) \delta^4(x-x_2) \delta^4(x-x_3)\end{aligned}$$

Therefore

$$\begin{aligned}\alpha(x, x_1, x_2, x_3) &= gf^{abc} g^{\mu\lambda} \partial_{x_1}^\nu \delta^4(x-x_1) \delta^4(x-x_2) \delta^4(x-x_3) \\ &= gf^{abc} g^{\mu\lambda} \partial_{x_1}^\nu \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4k_3}{(2\pi)^4} e^{-ik_1(x-x_1)} e^{-ik_2(x-x_2)} e^{-ik_3(x-x_3)} \\ &= \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4k_3}{(2\pi)^4} gf^{abc} g^{\mu\lambda} ik_1^\nu e^{-ik_1(x-x_1)} e^{-ik_2(x-x_2)} e^{-ik_3(x-x_3)}\end{aligned}$$

and

$$\tilde{\alpha}(k_1, k_2, k_3) = gf^{abc} g^{\mu\lambda} ik_1^\nu$$

Then

$$I = i \sum_{\text{permutations}} \tilde{\alpha}(k_1, k_2, k_3) = gf^{abc} [(k_2 - k_3)_\mu g_{\nu\lambda} + (k_3 - k_1)_\nu g_{\lambda\mu} + (k_1 - k_2)_\lambda g_{\mu\nu}]$$

9.6.3 List of Feynman rules

Propagators:

The propagators for the gauge bosons are in the Feynman gauge ($\xi = 1$).

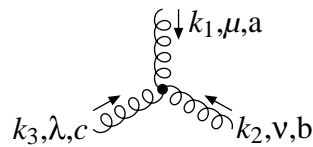
gauge bosons	gluon	A_μ^a	$\frac{-ig_{\mu\nu}}{k^2} \delta_{ab}$
	photon	A_μ	$\frac{-ig_{\mu\nu}}{k^2}$
fermions	quarks	ψ_i	$i \frac{\not{p} + m}{p^2 - m^2} \delta_{ij}$
	leptons	ψ	$i \frac{\not{p} + m}{p^2 - m^2}$
ghosts		c^a	$\frac{i}{k^2} \delta^{ab}$

Vertices:

Quark-gluon-vertex:

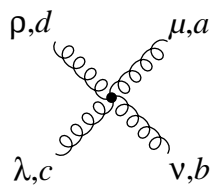
$$ig\gamma_\mu T_{ij}^a$$

3-gluon-vertex:



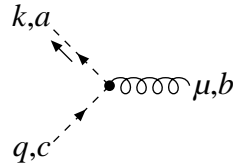
$$gf^{abc} [(k_2 - k_3)_\mu g_{\nu\lambda} + (k_3 - k_1)_\nu g_{\lambda\mu} + (k_1 - k_2)_\lambda g_{\mu\nu}]$$

4-gluon-vertex:



$$-ig^2 \left[f^{abe} f^{ecd} (g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda}) + f^{ace} f^{ebd} (g_{\mu\nu} g_{\lambda\rho} - g_{\mu\rho} g_{\lambda\nu}) + f^{ade} f^{ebc} (g_{\mu\nu} g_{\lambda\rho} - g_{\mu\lambda} g_{\nu\rho}) \right]$$

Gluon-ghost-vertex:



$$-g f^{abc} k_\mu$$

Fermion-photon-vertex:

$$ieQ\gamma_\mu$$

Additional rules:

An integration

$$\int \frac{d^4 k}{(2\pi)^4}$$

for each loop.

A factor (-1) for each closed fermion loop.

Symmetry factor: Multiply the diagram by a factor $1/S$, where S is the order of the permutation group of the internal lines and vertices leaving the diagram unchanged when the external lines are fixed.

External particles:

Outgoing fermion: $\bar{u}(p)$

Outgoing antifermion: $v(p)$

Incoming fermion: $u(p)$

Incoming antifermion: $\bar{v}(p)$

Gauge boson: $\epsilon_\mu(k)$

Polarization sums:

$$\begin{aligned}\sum_{\lambda} u(p, \lambda) \bar{u}(p, \lambda) &= \not{p} + m, \\ \sum_{\lambda} v(p, \lambda) \bar{v}(p, \lambda) &= \not{p} - m,\end{aligned}$$

$$\sum_{\lambda} \epsilon_{\mu}^*(k, \lambda) \epsilon_{\nu}(k, \lambda) = -g_{\mu\nu} + \frac{k_{\mu} n_{\nu} + n_{\mu} k_{\nu}}{kn} - n^2 \frac{k_{\mu} k_{\nu}}{(kn)^2}.$$

Here n^{μ} is an arbitrary four vector. The dependence on n^{μ} cancels in gauge-invariant quantities. Using Weyl spinors, a convenient choice of polarization vectors for the gauge bosons is given by

$$\begin{aligned}\epsilon_{\mu}^{+}(k, q) &= \frac{\langle q - |\gamma_{\mu}| k - \rangle}{\sqrt{2} \langle qk \rangle} \\ \epsilon_{\mu}^{-}(k, q) &= \frac{\langle q + |\gamma_{\mu}| k + \rangle}{\sqrt{2} [kq]}\end{aligned}$$

where q^{μ} is an arbitrary light-like reference momentum. The dependence on q^{μ} cancels in gauge-invariant quantities. The polarization vectors satisfy:

$$\begin{aligned}\epsilon_{\mu}^{\pm}(k, q) k^{\mu} &= 0, \\ \epsilon_{\mu}^{\pm}(k, q) q^{\mu} &= 0.\end{aligned}$$

$$\begin{aligned}\epsilon^{+} \cdot (\epsilon^{+})^{*} &= \epsilon^{-} \cdot (\epsilon^{-})^{*} = -1, \\ \epsilon^{+} \cdot (\epsilon^{-})^{*} &= 0.\end{aligned}$$

$$(\epsilon_{\mu}^{+})^{*} = \epsilon_{\mu}^{-}.$$

10 Spontaneously broken gauge theories

The concept of gauge theories allowed us to describe successfully quantum electrodynamics and quantum chromodynamics, the quantum theories of the electromagnetic and the strong force. Both theories are characterized by the fact, that the particles which mediate the forces (photons and gluons) are massless particles. This is required by gauge invariance. In fact, a naive mass term for the gauge bosons in the Lagrangian

$$\mathcal{L}_{mass} = m^2 A_\mu^a A^{a\mu}$$

is not invariant under gauge invariance

$$T^a A_\mu^a(x) \rightarrow U(x) \left(T^a A_\mu^a(x) + \frac{i}{g} \partial_\mu \right) U^\dagger(x).$$

On the other hand it is an experimental fact, that the W -bosons and the Z -boson have non-zero masses. As we do not want to abandon the concept of gauge theories, we face the problem on how to incorporate massive gauge bosons into gauge theories. The solution is provided by the concept of spontaneously broken gauge theories, also known under the name ‘‘Higgs mechanism’’. To start the discussion let us consider a simple physical system with a complex coordinate ϕ and a potential

$$V(\phi) = m^2 |\phi|^2 + \frac{1}{4} \lambda \left(|\phi|^2 \right)^2.$$

The potential has a harmonic term $m^2 |\phi|^2$ and an anharmonic term $\frac{1}{4} \lambda \left(|\phi|^2 \right)^2$. For $m^2 > 0$ and $\lambda > 0$ has an absolute minimum at $\phi = 0$. In classical mechanics the ground state would therefore be $\phi = 0$. This is nothing new.

Imagine now that the potential is given by

$$V(\phi) = -\mu^2 |\phi|^2 + \frac{1}{4} \lambda \left(|\phi|^2 \right)^2,$$

with

$$\mu^2 > 0, \quad \lambda > 0.$$

Then the potential has the shape of a mexican hat, and $\phi = 0$ corresponds to a local maxima. The potential has a minima for

$$|\phi|^2 = \frac{2\mu^2}{\lambda}.$$

The minimas are described by a circle in the complex plane. The ground state of the physical system will be one point of this circle, with no preference for any particular point. Without loss of generality we can choose this point to lie along the positive real axis. Therefore we face the situation, that the potential has a rotational symmetry around the point $\phi = 0$, while the ground state has not. This is the concept of a spontaneously broken symmetry. In general one speaks about a spontaneously broken symmetry, if the Lagrangian of a theory has a certain symmetry, which is not preserved in the ground state of the theory.

10.1 The Higgs boson

The standard model is based on the gauge group

$$SU(3) \otimes SU(2) \otimes U(1),$$

where $SU(3)$ is the gauge group of the strong interactions, $SU(2)$ the gauge group associated to the weak isospin and $U(1)$ the gauge group associated to the hypercharge. This group is not identical to the gauge group of quantum electrodynamics. To avoid confusions, one often writes $U_Y(1)$ for the group related to the hypercharge and $U_{el-magn}(1)$ for the gauge group of QED. In the standard model, the electroweak sector $SU(2) \otimes U_Y(1)$ is spontaneously broken down to $U_{el-magn}(1)$. We now study the spontaneously symmetry breakdown in detail.

Within the standard model one assumes an additional complex scalar field, transforming as the fundamental representation of $SU(2)$ and having hypercharge $Y = 1$. In the weak isospin space we can write the field as a two-vector with complex entries:

$$\phi(x) = \begin{pmatrix} \phi^+(x) \\ \frac{1}{\sqrt{2}}(v + H(x) + i\chi(x)) \end{pmatrix}.$$

$\phi^+(x)$ is a complex field (two real components). The three components $\phi(x)$ and $\chi(x)$ are absorbed as the longitudinal modes of W_μ^\pm and Z_μ . $H(x)$ is the Higgs field.

The Lagrange density of the Higgs sector

$$\mathcal{L}_{Higgs} = (D_\mu \phi)^\dagger (D^\mu \phi) - V(\phi) + \mathcal{L}_{Yukawa},$$

The covariant derivative is given by

$$D_\mu = \partial_\mu - igI^a W_\mu^a - ig' \frac{Y}{2} B_\mu$$

where $I^a = \frac{1}{2}\sigma^a$ (σ^a are the Pauli matrices) and we have $Y = 1$ for the Higgs doublet. (Note that our $g' = -g_1$ (Hollik).)

The Higgs potential is given by

$$V(\phi) = -\mu^2 \phi^\dagger \phi + \frac{1}{4} \lambda (\phi^\dagger \phi)^2$$

For $\mu^2 > 0$ (and $\lambda > 0$) we have spontaneous symmetry breaking. In that case the potential has a minimum for

$$\phi^\dagger \phi = \frac{2\mu^2}{\lambda} = \frac{v^2}{2}$$

We have

$$v = 2\sqrt{\frac{\mu^2}{\lambda}}$$

We write

$$\begin{aligned}\phi(x) &= \begin{pmatrix} \phi^+(x) \\ \frac{1}{\sqrt{2}}(v + H(x) + i\chi(x)) \end{pmatrix} \\ \phi^\dagger(x) &= \begin{pmatrix} \phi^-(x), \frac{1}{\sqrt{2}}(v + H(x) - i\chi(x)) \end{pmatrix}\end{aligned}$$

Then

$$\begin{aligned}(D_\mu\phi)^\dagger(D_\mu\phi)\Big|_{\text{physical}} &= \frac{1}{2}\partial_\mu H\partial_\mu H + \frac{1}{8}g^2v^2(W_\mu^1W_\mu^1 + W_\mu^2W_\mu^2) \\ &\quad + \frac{1}{8}v^2(B_\mu, W_\mu^3) \begin{pmatrix} g'^2 & -gg' \\ -gg' & g^2 \end{pmatrix} \begin{pmatrix} B_\mu \\ W_\mu^3 \end{pmatrix}\end{aligned}$$

The subscript ‘‘physical’’ indicates that we ignored contributions involving the $\phi^+(x)$ or $\chi(x)$ -fields. We define

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2)$$

and

$$\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} B_\mu \\ W_\mu^3 \end{pmatrix}$$

with

$$\cos\theta_W = \frac{g}{\sqrt{g^2 + g'^2}} \quad \sin\theta_W = \frac{g'}{\sqrt{g^2 + g'^2}}$$

We then obtain

$$\begin{aligned}(D_\mu\phi)^\dagger(D_\mu\phi)\Big|_{\text{physical}} &= \frac{1}{2}\partial_\mu H\partial_\mu H + \frac{1}{2}\left(\frac{vg}{2}\right)^2(W_\mu^{+*}W_\mu^+ + W_\mu^{-*}W_\mu^-) \\ &\quad + \frac{1}{2}\left(\frac{v}{2}\sqrt{g^2 + g'^2}\right)^2(A_\mu, Z_\mu) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix}\end{aligned}$$

We therefore have

$$m_W = \frac{v}{2}g \quad m_Z = \frac{v}{2}\sqrt{g^2 + g'^2}$$

It is not too difficult to show that the Higgs mass is given by

$$m_H = \frac{v}{2} \sqrt{\frac{\lambda}{2}}.$$

The gauge couplings g and g' are related to the elementary electric charge by

$$e = \frac{gg'}{\sqrt{g^2 + g'^2}},$$

or

$$g = \frac{e}{\sin \theta_W}, \quad g' = \frac{e}{\cos \theta_W}.$$

10.2 Yukawa couplings

The Higgs sector generates also the fermion masses through Yukawa couplings. We discuss this mechanism first in a simplified model without flavour mixing. The full standard model including flavour mixing is discussed in the next chapter.

Spin 1/2 particles are described by four-component spinors $\psi(x)$. With the chiral projectors

$$P_{\pm} = \frac{1}{2}(1 \pm \gamma_5)$$

we define left- and right-handed spinors:

$$\psi_{\pm}(x) = P_{\pm}\psi(x).$$

The fermions in the standard model can be grouped into three families:

$$\begin{pmatrix} u \\ d \\ \nu_e \\ e \end{pmatrix}, \quad \begin{pmatrix} c \\ s \\ \nu_{\mu} \\ \mu \end{pmatrix}, \quad \begin{pmatrix} t \\ b \\ \nu_{\tau} \\ \tau \end{pmatrix}.$$

The families differ only by the masses of their members.

The quantum numbers of the fermions in the electro-weak sector:

The left-handed components (u_L, d_L) and (ν_L, e_L) transform as the fundamental representation under the $SU(2)$ group. The right-handed components u_R, d_R, ν_R and e_R transform as a singlet under the $SU(2)$ group.

In detail one has:

	I_3	Y	Q		I_3	Y	Q
u_L	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	u_R	0	$\frac{4}{3}$	$\frac{2}{3}$
d_L	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{3}$	d_R	0	$-\frac{2}{3}$	$-\frac{1}{3}$
ν_L	$\frac{1}{2}$	-1	0	ν_R	0	0	0
e_L	$-\frac{1}{2}$	-1	-1	e_R	0	-2	-1

The electric charge is given by the Gell-Mann-Nishijima formula:

$$Q = I_3 + \frac{Y}{2}$$

Remark: The table contains a right-handed neutrino, which does not interact with any other particle.

The Yukawa couplings are given by

$$\mathcal{L}_{Yukawa} = \sum_{families} \left\{ -\lambda_d (\bar{u}_L, \bar{d}_L) \phi d_R - \lambda_u (\bar{u}_L, \bar{d}_L) \phi^C u_R - \lambda_e (\bar{\nu}_L, \bar{e}_L) \phi e_R - \lambda_\nu (\bar{\nu}_L, \bar{e}_L) \phi^C \nu_R + \text{h.c.} \right\}$$

where the charge-conjugate Higgs field is given by

$$\phi^C = 2iI^2 \phi^* = i\sigma^2 \phi^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \phi^-(x) \\ \frac{1}{\sqrt{2}}(v + H(x) - i\chi(x)) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(v + H(x) - i\chi(x)) \\ -\phi^-(x) \end{pmatrix}$$

Example:

$$\begin{aligned} -\lambda_d (\bar{u}_L, \bar{d}_L) \phi d_R + \text{h.c.} &= -\lambda_d (\bar{u}_L, \bar{d}_L) \begin{pmatrix} \phi^+(x) \\ \frac{1}{\sqrt{2}}(v + H(x) + i\chi(x)) \end{pmatrix} d_R + \text{h.c.} \\ &= -\frac{v\lambda_d}{\sqrt{2}} (\bar{u}_L, \bar{d}_L) \begin{pmatrix} 0 \\ 1 \end{pmatrix} d_R + \text{interaction terms} + \text{h.c.} \\ &= -\frac{v\lambda_d}{\sqrt{2}} \bar{d}_L d_R + \text{interaction terms} + \text{h.c.} \end{aligned}$$

Therefore the Yukawa couplings generate the masses of the fermions. From the above example we obtain

$$m_d = \frac{1}{\sqrt{2}} v \lambda_d.$$

10.3 Feynman rules in the electroweak sector

As with any gauge theory, we also have to fix the gauge for the electroweak sector. A useful gauge fixing condition is given in the electroweak sector by the 't Hooft gauge (also called R_ξ -gauge):

$$\mathcal{L}_{GF} = \underbrace{-\frac{1}{\xi_W}(\partial^\mu W_\mu^+ - im_W \xi_W \phi^+)(\partial^\mu W_\mu^- + im_W \xi_W \phi^-) - \frac{1}{2\xi_Z}(\partial^\mu Z_\mu - m_Z \xi_Z \chi)^2}_{SU(2)} - \underbrace{\frac{1}{2\xi_Y}(\partial^\mu A_\mu)^2}_{U(1)}$$

$\xi = 0$ corresponds to Landau gauge, $\xi = 1$ to the Feynman gauge. ϕ^+ , ϕ^- and χ are called the would-be Goldstone fields.

We now list the most important Feynman rules for the electroweak sector.

The propagators for the W - and Z -bosons are given in 't Hooft's $R_{\xi=1}$ gauge.

gauge bosons	photon	A_μ	$\frac{-ig_{\mu\nu}}{k^2}$
	W-boson	W_μ^\pm	$\frac{-ig_{\mu\nu}}{k^2 - m_W^2}$
	Z-boson	Z_μ	$\frac{-ig_{\mu\nu}}{k^2 - m_Z^2}$
Higgs sector	Higgs	H	$\frac{i}{k^2 - m_H^2}$

Fermion-Z-vertex:

$$\frac{ie}{2 \sin \theta_W \cos \theta_W} \gamma_\mu (v_f - a_f \gamma_5),$$

where

$$\begin{aligned} v_f &= I_3 - 2Q \sin^2 \theta_W, \\ a_f &= I_3. \end{aligned}$$

Fermion-W-vertex:

$$\frac{ie}{2\sqrt{2} \sin \theta_w} \gamma_\mu (1 - \gamma_5) V_{jk}$$

Fermion-Higgs-vertex:

$$\frac{ie}{2 \sin \theta_w} \frac{m_f}{m_W}$$

11 Flavour mixing

We now come to the final ingredient of the standard model: flavour mixing. Let us consider the electroweak sector, in particular the coupling of quarks to the electroweak gauge bosons. Recall the Lagrange density for the quark sector:

$$\mathcal{L}_{fermions} = \sum_{families} \left\{ (\bar{u}_L, \bar{d}'_L) i\gamma^\mu D_\mu \begin{pmatrix} u_L \\ d'_L \end{pmatrix} + \bar{u}_R i\gamma^\mu D_\mu u_R + \bar{d}'_R i\gamma^\mu D_\mu d'_R \right\}$$

For reasons, which will become clear later, we put a prime on all d -type quark fields. The Lagrange density is obtained by replacing the ordinary derivative ∂_μ with the covariant derivative

$$D_\mu = \partial_\mu - igI^a W_\mu^a - ig' \frac{Y}{2} B_\mu.$$

This is required by gauge invariance. This Lagrange density does not allow for mixing between the various quark flavours.

On the other hand the Yukawa couplings are given by

$$\mathcal{L}_{Yukawa} = \sum_{families} \left\{ -\lambda_d (\bar{u}_L, \bar{d}'_L) \phi d_R - \lambda_u (\bar{u}_L, \bar{d}'_L) \phi^C u_R + \text{h.c.} \right\}$$

where the charge-conjugate Higgs field is given by

$$\phi^C = i\sigma_2 \phi^* = 2iI_2 \phi^*.$$

Note that now the prime is missing on the d -type quark fields. We have seen that the Yukawa terms lead to mass terms for the fermions:

$$\begin{aligned} & -\lambda_d (\bar{u}_L, \bar{d}'_L) \phi d_R - \lambda_u (\bar{u}_L, \bar{d}'_L) \phi^C u_R + \text{h.c.} = \\ & -\frac{v\lambda_d}{\sqrt{2}} \bar{d}'_L d_R - \frac{v\lambda_u}{\sqrt{2}} \bar{u}_L u_R + \text{h.c.} + \text{interaction terms} \end{aligned}$$

However, the Yukawa couplings are not constrained by any gauge symmetry and we could allow for flavour mixing in the Yukawa terms. In fact, nature has chosen this possibility. We therefore consider a general mass term of the form

$$\mathcal{L}_{Mass} = \sum_{families} \bar{d}'_L M_d d''_R + \bar{u}'_L M_u u''_R + \text{h.c.}$$

where M_d and M_u are (arbitrary) complex 3×3 matrices in family space. A matrix M can be diagonalized by a biunitary transformation

$$V^{-1} M W = \tilde{M},$$

where \tilde{M} is a diagonal matrix.

Proof: Using the polar decomposition, M can be written as

$$M = HU,$$

where H is hermitian and U is a unitary matrix. H can be diagonalized by a unitary matrix V :

$$V^{-1}HV = \tilde{M},$$

therefore $W = U^{-1}V$.

The gauge part of the Lagrange density

$$\mathcal{L}_{fermions} = \sum_{families} \left\{ (\bar{u}_L, \bar{d}_L) i\gamma^\mu D_\mu \begin{pmatrix} u_L \\ d'_L \end{pmatrix} + \bar{u}_R i\gamma^\mu D_\mu u_R + \bar{d}_R i\gamma^\mu D_\mu d'_R \right\}$$

is invariant under the rotations with respect to the family index:

$$\begin{aligned} \begin{pmatrix} u_L \\ d'_L \end{pmatrix} &\rightarrow S_L \begin{pmatrix} u_L \\ d'_L \end{pmatrix} \\ u_R &\rightarrow S_{R,u} u_R \\ d_R &\rightarrow S_{R,d} d_R \end{aligned}$$

Using this freedom we have with

$$\begin{aligned} M_u &= V_u \tilde{M}_u W_u^{-1}, \quad M_d = V_d \tilde{M}_d W_d^{-1}, \\ \sum_{families} \underbrace{\bar{d}'_L V_d}_{\bar{d}'_L V_u^{-1} V_d} \underbrace{\tilde{M}_d W_d^{-1} d''_R}_{d'_R} + \underbrace{\bar{u}'_L V_u}_{\bar{u}'_L} \underbrace{\tilde{M}_u W_u^{-1} u''_R}_{u'_R} + \text{h.c.} &= \\ \sum_{families} \underbrace{\bar{d}'_L V_u^{-1} V_d}_{\bar{d}'_L} \tilde{M}_d d'_R + \bar{u}'_L \tilde{M}_u u'_R + \text{h.c.} & \end{aligned}$$

$V_u^{-1}V_d$ describes the quark mixing and is a unitary 3×3 matrix:

$$V_{CKM} = V_u^{-1}V_d.$$

Note that

$$d' = V_{CKM}d \quad \text{and} \quad \bar{d}' V_{CKM} = \bar{d}.$$

A unitary $n \times n$ matrix is described by n^2 real parameters, out of these

$$\frac{n(n+1)}{2}$$

are phases. For 3×3 matrices we have three angles and six phases. We still have the freedom to redefine our fields by a unitary diagonal matrix:

$$\Psi \rightarrow \begin{pmatrix} e^{i\phi_1} & 0 & 0 \\ 0 & e^{i\phi_2} & 0 \\ 0 & 0 & e^{i\phi_3} \end{pmatrix} \Psi$$

This can be used to eliminate $2n - 1$ phases, e.g. five out of the six phases for three generations. This leaves one “physical” phase in the CKM-matrix.

Standard parameterizations:

The CKM-matrix connects the weak eigenstates (d', s', b') with the mass eigenstates (d, s, b) :

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

Standard parametrization:

$$V_{CKM} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -s_{23}c_{12} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix},$$

with $c_{ij} = \cos \theta_{ij}$ and $s_{ij} = \sin \theta_{ij}$.

Wolfenstein parametrization:

$$V_{CKM} = \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + O(\lambda^4).$$

Neutrino mixing:

In the lepton sector one uses for Dirac neutrinos the lepton mixing matrix

$$\begin{pmatrix} \nu'_e \\ \nu'_\mu \\ \nu'_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

ν'_e , ν'_μ and ν'_τ are the weak eigenstates, whereas ν_1 , ν_2 and ν_3 are the mass eigenstates.

12 Summary: The Standard Model

To conclude these lectures we summarize the Lagrange density of the standard model. The Lagrange density for the standard model is split into three parts:

$$\mathcal{L}_{SM} = \mathcal{L}_{gauge} + \mathcal{L}_{fermions} + \mathcal{L}_{Higgs}.$$

The Lagrange density for the gauge bosons:

$$\mathcal{L}_{gauge} = \underbrace{-\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu}}_{SU(3)} - \underbrace{\frac{1}{4}W_{\mu\nu}^a W^{\mu\nu a}}_{SU(2)} - \underbrace{\frac{1}{4}B_{\mu\nu} B^{\mu\nu}}_{U(1)} + \mathcal{L}_{GF} + \mathcal{L}_{FP}$$

where

$$\begin{aligned} F_{\mu\nu}^a &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_3 f_{SU(3)}^{abc} A_\mu^b A_\nu^c, \\ W_{\mu\nu}^a &= \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g f_{SU(2)}^{abc} W_\mu^b W_\nu^c \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \end{aligned}$$

For $SU(3)$, the indices a, b and c label the generators of $SU(3)$ and run from 1 to 8. For $SU(2)$, they label the generators of $SU(2)$ and run from 1 to 3. The gauge fixing part ('t Hooft gauge):

$$\begin{aligned} \mathcal{L}_{GF} &= \underbrace{-\frac{1}{2\xi_g}(\partial^\mu A_\mu^a)^2}_{SU(3)} \\ &\quad - \underbrace{\frac{1}{\xi_W}(\partial^\mu W_\mu^+ - im_W \xi_W \phi^+)(\partial^\mu W_\mu^- + im_W \xi_W \phi^-) - \frac{1}{2\xi_Z}(\partial^\mu Z_\mu - m_Z \xi_Z \chi)^2}_{SU(2)} - \underbrace{\frac{1}{2\xi_\gamma}(\partial^\mu A_\mu)^2}_{U(1)} \end{aligned}$$

$\xi = 0$ corresponds to Landau gauge, $\xi = 1$ to the Feynman gauge. ϕ^+, ϕ^- and χ are called the would-be Goldstone fields and have their origin in the Higgs sector. The fields W_μ^a and B_μ are related to the W_μ^\pm, Z_μ and A_μ fields as follows:

$$\begin{aligned} W_\mu^\pm &= \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2), \\ \begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} &= \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} B_\mu \\ W_\mu^3 \end{pmatrix}. \end{aligned}$$

The Faddeev-Popov term for QCD reads:

$$\mathcal{L}_{FP} = \bar{c}^a(x) \left(-\partial^\mu D_\mu^{ab} \right) c^b(x)$$

The covariant derivative in the fundamental representation reads

$$D_\mu^{ab} = \delta^{ab} \partial_\mu - g_3 f_{SU(3)}^{abc} A_\mu^c.$$

In the electroweak sector we have the ghost fields $d^\pm(x)$, $d^Z(x)$ and $d^Y(x)$. The Faddeev-Popov term has the form

$$\mathcal{L}_{FP} = \bar{d}^{\alpha}(x) K^{\alpha\beta} d^{\beta}(x)$$

The Lagrange density for the fermion sector:

$$\begin{aligned} \mathcal{L}_{fermions} = & \sum_{families} \left\{ (\bar{u}_L, \bar{d}'_L) i\gamma^\mu D_\mu \begin{pmatrix} u_L \\ d'_L \end{pmatrix} + \bar{u}_R i\gamma^\mu D_\mu u_R + \bar{d}'_R i\gamma^\mu D_\mu d'_R \right. \\ & \left. + (\bar{\nu}'_L, \bar{e}_L) i\gamma^\mu D_\mu \begin{pmatrix} \nu'_L \\ e_L \end{pmatrix} + \bar{\nu}'_R i\gamma^\mu D_\mu \nu'_R + \bar{e}_R i\gamma^\mu D_\mu e_R \right\} \\ D_\mu = & \begin{cases} \partial_\mu - igT^a A_\mu^a - igI^a W_\mu^a - ig' \frac{Y}{2} B_\mu & \text{quarks} \\ \partial_\mu - igI^a W_\mu^a - ig' \frac{Y}{2} B_\mu & \text{leptons} \end{cases} \end{aligned}$$

Note that a right-handed neutrino (with no interactions through D_μ) has been added.

The Lagrange density of the Higgs sector

$$\mathcal{L}_{Higgs} = (D_\mu \phi)^\dagger (D^\mu \phi) + \mu^2 \phi^\dagger \phi - \frac{1}{4} \lambda (\phi^\dagger \phi)^2 + \mathcal{L}_{Yukawa},$$

The covariant derivative is given as before by

$$D_\mu = \partial_\mu - igI^a W_\mu^a - ig' \frac{Y}{2} B_\mu.$$

The Higgs doublet is parameterized as follows:

$$\begin{aligned} \phi(x) &= \begin{pmatrix} \phi^+(x) \\ \frac{1}{\sqrt{2}}(v + H(x) + i\chi(x)) \end{pmatrix} \\ \phi^\dagger(x) &= \begin{pmatrix} \phi^-(x), \frac{1}{\sqrt{2}}(v + H(x) - i\chi(x)) \end{pmatrix} \end{aligned}$$

The Higgs doublet has $Y = 1$.

The Yukawa couplings are given by

$$\mathcal{L}_{Yukawa} = \sum_{families} \left\{ -\lambda_d (\bar{u}_L, \bar{d}'_L) \phi d_R - \lambda_u (\bar{u}_L, \bar{d}'_L) \phi^C u_R - \lambda_e (\bar{\nu}'_L, \bar{e}_L) \phi e_R - \lambda_\nu (\bar{\nu}'_L, \bar{e}_L) \phi^C \nu_R + \text{h.c.} \right\}$$

The CKM-matrix connects the weak eigenstates (d', s', b') with the mass eigenstates (d, s, b):

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

In the lepton sector one uses for Dirac neutrinos the lepton mixing matrix

$$\begin{pmatrix} \nu'_e \\ \nu'_\mu \\ \nu'_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

ν'_e, ν'_μ and ν'_τ are the weak eigenstates, whereas ν_1, ν_2 and ν_3 are the mass eigenstates.