Lie groups

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- Classifikation of semi-simple groups

1.2 Literature

- Introductory texts:

  - W. Fulton and J. Harris, Representation theory, Springer, 1991
  - B. Hall, Lie groups, Lie Algebras and Representations, Springer, 2003

- Physics related:

  - M. Schottenloher, Geometrie und Symmetrie in der Physik, Vieweg, 1995
  - M. Nakahara, Geometry, Topology and Physics, IOP, 1990

- Classics:

  - N. Bourbaki, Groupes et algèbres de Lie, Hermann, 1972
  - H. Weyl, The classical groups, Princeton University Press, 1946

- Differential geometry:

  - S. Helgason, Differential Geometry, Lie Groups and Symmetric Spaces, AMS, 1978

- Hopf algebras:

  - Ch. Kassel, Quantum Groups, Springer, 1995

- Specialised topics:

  - Ch. Reutenauer, Free Lie Algebras, Clarendon Press, 1993
  - V. Kac, Infinite dimensional Lie algebras, Cambridge University Press, 1983

2 Basics

2.1 Groups

2.1.1 Definition

A non-empty set G together with a composition $\cdot : G \times G \to G$ is called a group $(G, \cdot)$ if

G1: The composition $\cdot$ is associative : $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

G2: There exists a neutral element : $e \cdot a = a \cdot e = a$ for all $a \in G$

G3: For all $a \in G$ there exists an inverse a^{-1} : $a^{-1} \cdot a = a \cdot a^{-1} = e$

One can actually use a weaker system of axioms:

G1': The composition $\cdot$ is associative : $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

G2': There exists a left-neutral element : $e \cdot a = a$ for all $a \in G$

G3': For all $a \in G$ there exists an left-inverse a^{-1} : $a^{-1} \cdot a = e$

The first system of axioms clearly implies the second system of axioms. To show that the second system also implies the first one, we show the following:
a) If e is a left-neutral element, and e' is a right-neutral element, then $e = e'$.
Proof:

$$\begin{aligned} e' &= e \cdot e' && e \text{ is left-neutral} \\ &= e, && e' \text{ is right-neutral} \end{aligned}$$

b) If b is a left-inverse to a, and b' is a right-inverse to a, then $b = b'$.
Proof:

$$\begin{aligned} b &= b \cdot e && e \text{ is right-neutral} \\ &= b \cdot \left(a \cdot b'\right) && b' \text{ is right-inverse of } a \\ &= (b \cdot a) \cdot b' && \text{associativity} \\ &= e \cdot b' && b \text{ is left-inverse of } a \\ &= b' && e \text{ is left-neutral} \end{aligned}$$

c) If b is a left-inverse to a, i.e. $b \cdot a = e$, then b is also the right-inverse to a.
Proof:

$$\begin{aligned} (a \cdot b) \cdot (a \cdot b) &= a \cdot (b \cdot a) \cdot b \\ &= a \cdot e \cdot b \\ &= a \cdot b \end{aligned}$$

Therefore $a \cdot b = e$.
d) If e is the left-neutral element, then e is also right-neutral.
Proof:

$$\begin{aligned} a &= e \cdot a \\ &= \left(a^{-1} \cdot a\right) \cdot a \\ &= \left(a \cdot a^{-1}\right) \cdot a \\ &= a \cdot \left(a^{-1} \cdot a\right) \\ &= a \cdot e \end{aligned}$$

This completes the proof that the second system of axioms is equivalent to the first system of axioms. To verify that a given set together with a given composition forms a group it is therefore sufficient to verify axioms (G2') and (G3') instead of axioms (G2) and (G3).

More definitions:
A group $(G, \cdot)$ is called Abelian if the operation $\cdot$ is commutative : $a \cdot b = b \cdot a$.

The number of elements in the set G is called the order of the group. If this number is finite, we speak of a finite group. In the case where the order is infinite, we can further distinguish the case where the set is countable or not. For Lie groups we are in particular interested in the latter case.

2.1.2 Examples

a) The trivial example: Let $G = \{e\}$ and $e \cdot e = e$. This is a group with one element.

b) $\mathbb{Z}_2$: Let $G = \{0, 1\}$ and denote the composition by $+$. The composition is given by the following composition table:

$+$	0	1
0	0	1
1	1	0

$\mathbb{Z}_2$ is of order 2 and is Abelian.

c) $\mathbb{Z}_n$: We can generalise the above example and take $G = \{0, 1, 2, ..., n-1\}$. We define the addition by

$$a + b \quad = \quad a + b \bmod n,$$

where on the l.h.s. "+" denotes the composition in $\mathbb{Z}_n$, whereas on the r.h.s. "+" denotes the usual addition of integer numbers. $\mathbb{Z}_n$ is a group of order n and is Abelian.

d) The symmetric group S_n: Let X be a set with distinct n elements and set

$$G = \{\sigma | \sigma : X \to X \text{ permutation of } X\}$$

As composition law we take the composition of permutations. The symmetric group has order

$$|S_n| = n!$$

For $n \geq 3$ this group is non-Abelian:

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

e) $(\mathbb{Z}, +)$: The integer numbers with addition form an Abelian group. The order of the group is infinite, but countable.

f) $(\mathbb{R}, +)$: The real numbers with addition form an Abelian group. The order of the group is not countable.

g) $(\mathbb{R}^*, \cdot)$: Denote by $\mathbb{R}^* = \mathbb{R} \backslash \{0\}$ the real numbers without zero. The set $\mathbb{R}^*$ with the multiplication as composition law forms an Abelian group.

h) Rotations in two dimensions: Consider the set of 2×2-matrixes

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix},$$

together with matrix multiplication as composition. To check this, one has to show that

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \cdot \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

can again be written as

$$\begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix}.$$

Using the addition theorems of sin and cos one finds $\gamma = \alpha + \beta$. The elements of this group are not countable, but they form a compact set.

2.1.3 Morphisms

Let a and b be elements of a group $(G,*)$ with composition $*$ and let a' and b' be elements of a group $(G,\circ)$ with composition $\circ$. We are interested in mappings between groups which preserve the structure of the compositions.

Homomorphism: We call a mapping $f : G \to G'$ a homomorphism, if

$$f(a*b) \quad = \quad f(a)\circ f(b).$$

Isomorphism: We call a mapping $f : G \to G'$ an isomorphism, if it is bijective and a homomorphism.

Automorphism: We call a mapping $f : G \to G$ from the group G into the group G itself an automorphism, if it is an isomorphism.

2.2 Manifolds

2.2.1 Definition

A **topological space** is a set M together with a family $\mathcal{T}$ of subsets of M satisfying the following properties:

1. $\emptyset \in \mathcal{T}, M \in \mathcal{T}$
2. $U_1, U_2 \in \mathcal{T} \Rightarrow U_1 \cap U_2 \in \mathcal{T}$
3. For any index set A we have $U_\alpha \in \mathcal{T}; \alpha \in A \Rightarrow \bigcup\limits_{\alpha\in A} U_\alpha \in \mathcal{T}$

The sets $U \in \mathcal{T}$ are called **open**.

A topological space is called **Hausdorff** if for any two distinct points $p_1, p_2 \in M$ there exists open sets $U_1, U_2 \in \mathcal{T}$ with

$$p_1 \in U_1, \quad p_2 \in U_2, \quad U_1 \cap U_2 = \emptyset.$$

A map between topological spaces is called **continous** if the preimage of any open set is again open.

A bijective map which is continous in both directions is called a **homeomorphism**.

An **open chart** on M is a pair (U,φ), where U is an open subset of M and φ is a homeomorphism of U onto an open subset of $\mathbb{R}^n$.

A **differentiable manifold** of dimension n is a Hausdorff space with a collection of open charts $(U_\alpha, \varphi_\alpha)_{\alpha\in A}$ such that

M1:

$$M = \bigcup_{\alpha \in A} U_\alpha.$$

M2: For each pair $\alpha, \beta \in A$ the mapping $\varphi_\beta \circ \varphi_\alpha^{-1}$ is an infinitely differentiable mapping of $\varphi_\alpha \left(U_\alpha \cap U_\beta\right)$ onto $\varphi_\beta \left(U_\alpha \cap U_\beta\right)$.

A differentiable manifold is also often denoted as a C^∞ manifold. As we will only be concerned with differentiable manifolds, we will often omitt the word "differentiable" and just speak about manifolds.

The collection of open charts $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ is called an **atlas**.

If $p \in U_\alpha$ and

$$\varphi_\alpha(p) = (x_1(p), ..., x_n(p)),$$

the set U_α is called the **coordinate neighbourhood** of p and the numbers $x_i(p)$ are called the **local coordinates** of p.

Note that in each coordinate neighbourhood M looks like an open subset of $\mathbb{R}^n$. But note that we do not require that M be $\mathbb{R}^n$ globally.

Consider two manifolds M and N with dimensions m and n. Let x_i be coordinates on M and y_j be coordinates on N. A mapping $f : M \to N$ between two manifolds is called **analytic**, if for each point $p \in M$ there exits a neighbourhood U of p and n power series P_j, $j = 1, ..., n$ such that

$$y_j(f(q)) = P_j\left(x_1(q) - x_1(p), ..., x_m(q) - x_m(p)\right)$$

for all $q \in U$.

An **analytic manifold** is a manifold where the mapping $\varphi_\beta \circ \varphi_\alpha^{-1}$ is analytic.

2.2.2 Examples

a) $\mathbb{R}^n$: The space $\mathbb{R}^n$ is a manifold. $\mathbb{R}^n$ can be covered with a single chart.

b) S^1: The circle

$$S^1 = \{\vec{x} \in \mathbb{R}^2 | |\vec{x}|^2 = 1\}$$

is a manifold. For an atlas we need at least two charts.

c) The set of rotation matrices in two dimensions:

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix},$$

The set of all these matrices forms a manifold homeomorphic to the circle S^1.

2.2.3 Morphisms

Homeomorphism: A map $f : M \to N$ between two manifolds M and N is called a homeomorphism if it is bijective and both the mapping $f : M \to N$ and the inverse $f^{-1} : N \to M$ are continous.

Diffeomorphism: A map $f : M \to N$ is called a diffeomorphism if it is a homeomorphism and both f and f^{-1} are infinitely differentiable.

Analytic diffeomorphism: The map $f : M \to N$ is a diffeomorphism and analytic.

2.3 Lie groups

2.3.1 Definition

A Lie group G is a group which is also an analytic manifold, such that the mappings

$$\begin{array}{rcl} G \times G & \to & G, \\ (a,b) & \to & a \cdot b, \end{array}$$

and

$$\begin{array}{rcl} G & \to & G, \\ a & \to & a^{-1} \end{array}$$

are analytic.

Remark: Instead of the two mappings above, it is sufficient to require that the mapping

$$\begin{array}{rcl} G \times G & \to & G, \\ (a,b) & \to & a \cdot b^{-1} \end{array}$$

is analytic.

2.3.2 Examples

The most important examples of Lie groups are matrix groups with matrix multiplication as composition. In order to have an inverse, the matrices must be non-singular.

a) $GL(n,\mathbb{R})$, $GL(n,\mathbb{C})$: The group of non-singular $n \times n$ matrices with real or complex entries. $GL(n,\mathbb{R})$ has n^2 real parameters, $GL(n,\mathbb{C})$ has $2n^2$ real parameters.

b) $SL(n,\mathbb{R})$, $SL(n,\mathbb{C})$: The group of non-singular $n \times n$ matrices with real or complex entries and

$$\det A = 1.$$

$SL(n,\mathbb{R})$ has n^2-1 real parameters, while $SL(n,\mathbb{C})$ has $2(n^2-1)$ real parameters.

c) $O(n)$: The group of orthogonal $n\times n$ matrices defined through

$$RR^T = 1.$$

The group $O(n)$ has $n(n-1)/2$ real parameters. The group $O(n)$ can also be defined as the transformation group of a real n-dimensional vector space, which preserves the inner product

$$\sum_{i=1}^{n} x_i^2$$

d) $SO(n)$: The group of special orthogonal $n\times n$ matrices defined through

$$RR^T = 1 \quad \text{and} \quad \det R = 1.$$

The group $SO(n)$ has $n(n-1)/2$ real parameters.

e) $U(n)$: The group of unitary $n\times n$ matrices defined through

$$UU^\dagger = 1.$$

The group $U(n)$ has n^2 real parameters. The group $U(n)$ can also be defined as the transformation group of a complex n-dimensional vector space, which preserves the inner product

$$\sum_{i=1}^{n} z_i^* z_i$$

f) $SU(n)$: The group of special unitary $n\times n$ matrices defined through

$$UU^\dagger = 1 \quad \text{and} \quad \det U = 1.$$

The group $SU(n)$ has n^2-1 real parameters.

g) $Sp(n,\mathbb{R})$: The symplectic group is the group of $2n\times 2n$ matrices satisfying

$$M^T \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} M = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

The group $Sp(n,\mathbb{R})$ has $(2n+1)n$ real parameters. The group $Sp(n,\mathbb{R})$ can also be defined as the transformation group of a real $2n$-dimensional vector space, which preserves the inner product

$$\sum_{j=1}^{n} \left(x_j y_{j+n} - x_{j+n} y_j \right).$$

2.4 Algebras

2.4.1 Definition

Let K be a field and A a vector space over the field K. A is called an algebra, if there is an additional composition

$$\begin{array}{rcl} A \times A & \to & A \\ (a_1, a_2) & \to & a_1 a_2 \end{array}$$

such that the algebra multiplication is K-linear.

$$\begin{array}{rcl} (r_1 a_1 + r_2 a_2) a_3 & = & r_1 (a_1 a_3) + r_2 (a_2 a_3) \\ a_3 (r_1 a_1 + r_2 a_2) & = & r_1 (a_3 a_1) + r_2 (a_3 a_2) \end{array}$$

Remark: It is not necessary to require that K is a field. It is sufficient to have a commutative ring R with 1. In this case one replaces the requirement for A to be a vector space by the requirement that A is an unital R-modul. The difference between a field K and a commutative ring R with 1 lies in the fact that in the ring R the multiplicative inverse might not exist.

An algebra is called associative if

$$(a_1 a_2) a_3 = a_1 (a_2 a_3)$$

An algebra is called commutative if

$$a_1 a_2 = a_2 a_1$$

An unit element $\mathbf{1}_A \in A$ satisfies

$$\mathbf{1}_A \, a = a.$$

Note that it is not required that A has a unit element. If there is one, note that difference between $\mathbf{1}_A \in A$ and $1_K \in K$: The latter always exists and we have the scalar multiplication with one:

$$1_K \, a = a.$$

2.4.2 Examples

a) Consider the set of $n \times n$ matrices over $\mathbb{R}$ with the composition given by matrix multiplication. This gives an associative, non-commutative algebra with a unit element given by the unit matrix.

b) Consider the set of $n \times n$ matrices over $\mathbb{R}$ where the composition is defined by

$$[a, b] = ab - ba.$$

This defines a non-associative, non-commutative algebra. There is no unit element.

2.5 Lie algebras

2.5.1 Definition

For a Lie algebra it is common practice to denote the composition of two elements a and b by $[a,b]$. An algebra is called a Lie-algebra if the composition satisfies

$$\begin{aligned} [a,a] &= 0, \\ [a,[b,c]]+[b,[c,a]]+[c,[a,b]] &= 0. \end{aligned}$$

Remark: Consider again the example above of the set of $n \times n$ matrices over $\mathbb{R}$ where the composition is defined by the commutator

$$[a,b] = ab - ba.$$

Clearly this definition satisfies $[a,a] = 0$. It fullfills the Jacobi identity:

$$\begin{aligned} & [a,[b,c]]+[b,[c,a]]+[c,[a,b]] = \\ &= abc - acb - bca + cba + bca - bac - cab + acb + cab - cba - abc + bac \\ &= 0. \end{aligned}$$

Matrix algebras with the commutator as composition are therefore Lie algebras.

Let A be a Lie algebra and $X_1, ..., X_n$ a basis of A as a vector space. $[X_i, X_j]$ is again in A and can be expressed as a linear combination of the basis vectors X_k:

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ijk} X_k.$$

The coefficents c_{ijk} are called the structure constants of the Lie algebra. For matrix algebras the X_i's are anti-hermitian matrices.

The notation above is mainly used in the mathematical literature. In physics a slightly different convention is often used: Denote by $T_1, ..., T_n$ a basis of A as a vector space. Then

$$[T_a, T_b] = i \sum_{c=1}^{n} f_{abc} T_c.$$

For matrix algebras the T_a's are hermitian matrices.

2.5.2 The exponential map

In this section we focus on matrix Lie groups. Let us first define the matrix exponential. For an $n \times n$ matrix X we define $\exp X$ by

$$\exp X = \sum_{n=0}^{\infty} \frac{X^n}{n!}.$$

Theorem: For any $n \times n$ real or complex matrix X the series converges.

A few properties:
1. We have

$$\exp(0) = 1.$$

2. $\exp X$ is invertible and

$$(\exp X)^{-1} = \exp(-X).$$

3. We have

$$\exp[(\alpha+\beta)X] = \exp(\alpha X)\exp(\beta X).$$

4. If $XY = YX$ then

$$\exp(X+Y) \quad = \quad \exp X \exp Y.$$

5. If A is invertible then

$$\exp\left(AXA^{-1}\right) \quad = \quad A\exp(X)A^{-1}.$$

6. We have

$$\frac{d}{dt}\exp(tX) \quad = \quad X\exp(tX) = \exp(tX)X.$$

In particular

$$\left.\frac{d}{dt}\exp(tX)\right|_{t=0} \quad = \quad X.$$

Point 1 is obvious. Points 2 and 3 are special cases of 4. To prove point 4 it is essential that X and Y commute:

$$\begin{aligned}
\exp X \exp Y &= \sum_{i=0}^{\infty}\frac{X^i}{i!}\sum_{j=0}^{\infty}\frac{Y^j}{j!} = \sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{X^i}{i!}\frac{Y^{n-i}}{(n-i)!} = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{i=0}^{n}\begin{pmatrix} n \\ i \end{pmatrix} X^i Y^{n-i} \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}(X+Y)^n = \exp(X+Y).
\end{aligned}$$

Proof of point 5:

$$\exp\left(AXA^{-1}\right) \quad = \quad \sum_{n=0}^{\infty}\frac{1}{n!}\left(AXA^{-1}\right)^n = \sum_{n=0}^{\infty}\frac{1}{n!}AX^nA^{-1} = A\exp(X)A^{-1}.$$

Proof of point 6:

$$\frac{d}{dt}\exp(tX) = \frac{d}{dt}\sum_{n=0}^{\infty}\frac{1}{n!}(tX)^n = \sum_{n=0}^{\infty}\frac{1}{(n-1)!}t^{n-1}X^n = X\exp(tX) = \exp(tX)X.$$

Computation of the exponential of a matrix:

Case 1: X is diagonalisable.

If $X = ADA^{-1}$ with $D = \text{diag}(\lambda_1, \lambda_2, ...)$ we have

$$\exp X = \exp ADA^{-1} = A\exp(D)A^{-1} = A \text{ diag}\left(e^{\lambda_1}, e^{\lambda_2}, ...\right) A^{-1}.$$

Case 2: X is nilpotent.

A matrix X is called nilpotent, if $X^m = 0$ for some positive m. In this case the series terminates:

$$\exp X = \sum_{n=0}^{m-1}\frac{X^n}{n!}$$

Case 3: X is arbitrary.

A general matrix X may be neither diagonalisable nor nilpotent. However, any matrix X can uniquely be written as

$$X = S + N,$$

where S is diagonalisable and N is nilpotent and $SN = NS$. Then

$$\exp X = \exp S \exp N$$

and $\exp S$ and $\exp N$ can be computed as in the previous two cases.

2.5.3 Relation between Lie algebras and Lie groups

Let G be a Lie group. Assume that as a manifold it has dimension n. G is also a group. Choose a local coordinate system, such that the identity element e is given by

$$e = g(0, ..., 0).$$

A lot of information on G can be obtained from the study of G in the neighbourhood of e. Let

$$g(\theta_1, ..., \theta_n)$$

denote a general point in the local chart containing e. Let us write

$$\begin{aligned} g(0,...,\theta_a,...,0) &= g(0,...,0,...,0) + \theta_a X^a + O(\theta^2) \\ &= g(0,...,0,...,0) + i\theta_a T^a + O(\theta^2). \end{aligned}$$

We also have

$$\begin{aligned} X^a &= \lim_{\theta_a \to 0} \frac{g(0,...,\theta_a,...,0) - g(0,...,0,...,0)}{\theta_a}, \\ T^a &= \lim_{\theta_a \to 0} \frac{g(0,...,\theta_a,...,0) - g(0,...,0,...,0)}{i\theta_a}. \end{aligned}$$

The T^a's are called the **generators** of the Lie group G.

Theorem: The commutators of the generators T^a of a Lie group are linear combinations of the generators and satisfy a Lie algebra.

$$\left[T^a, T^b\right] = i \sum_{c=1}^{n} f^{abc} T^c.$$

We will often use Einstein's summation convention and simply write

$$\left[T^a, T^b\right] = i f^{abc} T^c.$$

In order to proove this theorem we have to show that the commutator is again a linear combination of the generators. We start with the definition of a one-parameter subgroup of $GL(n,\mathbb{C})$: A map $g : \mathbb{R} \to GL(n,\mathbb{C})$ is called a one-parameter sub-group of $GL(n,\mathbb{C})$ if
1. $g(t)$ is continous.

2. $g(0) = 1$.

3. For $t_1, t_2 \in \mathbb{R}$ we have

$$g(t_1 + t_2) = g(t_1)\, g(t_2).$$

If $g(t)$ is a one-parameter sub-group of $GL(n,\mathbb{C})$ then there exists a unique $n \times n$ matrix X such that

$$g(t) = \exp(tX).$$

X is given by

$$X = \left. \frac{d}{dt} g(t) \right|_{t=0}.$$

There is a one-to-one correspondence between linear combinations of the generators

$$X = i\theta_a T^a$$

and the one-parameter sub-groups

$$g(t) = \exp(tX) \quad \text{with} \quad X = \left. \frac{d}{dt} g(t) \right|_{t=0}.$$

If $A \in G$ and if Y defines a one-parameter sub-group of G, then also AYA^{-1} defines a one-parameter sub-group of G. The non-trivial point is to check that $\exp\left[t\left(AYA^{-1}\right)\right]$ is again in G. This follows from

$$\exp\left[t\left(AYA^{-1}\right)\right] = A\exp(tY)A^{-1}.$$

Therefore AYA^{-1} is a linear combination of the generators. Now we take for $A = \exp(\lambda X)$. This implies that

$$\exp(\lambda X) Y \exp(-\lambda X)$$

is a linear combination of the generators. Since the vector space spanned by the generators is topologically closed, also the derivative with respect to λ belongs to this vector space and we have shown that

$$\left. \frac{d}{d\lambda} \exp(\lambda X) Y \exp(-\lambda X) \right|_{\lambda=0} = XY - YX = [X,Y]$$

is again a linear combination of the generators.

We have seen that by studying a Lie group G in the neighbourhood of the identity we can obtain from the Lie group G the corresponding Lie algebra $\mathfrak{g}$. We can now ask if the converse is also true: Given the Lie algebra $\mathfrak{g}$, can we reconstruct the Lie group G ? The answer is that this can almost be done. Note that a Lie group need not be connected. The Lorentz group is an example of a Lie group which is not connected. Given a Lie algebra we have information about the connected component in which the idenity lies. The exponential map takes us from the Lie algebra into the group. In the neighbourhood of the identity we have

$$g(\theta_1, ..., \theta_n) = \exp\left(i \sum_{a=1}^{n} \theta_a T^a\right).$$

2.5.4 Examples

As an example for the generators of a group let us study the cases of $SU(2)$ and $SU(3)$, as well as the groups $U(2)$ and $U(3)$. A common normalisation for the generators is

$$\text{Tr}\, T^a T^b = \frac{1}{2}\delta^{ab}.$$

a) The group $SU(2)$ is a three-paramter group. The generators are proportional to the Pauli matrices:

$$T^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T^2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

b) The group $SU(3)$ has eight parameters. The generators can be taken as the Gell-Mann matrices:

$$T^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$T^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad T^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$T^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

c) For the groups $U(2)$ and $U(3)$ add the generator

$$T^0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for $U(2)$, respectively the generator

$$T^0 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for $U(3)$.

2.5.5 The Fierz identity

Problem: Denote by T^a the generators of $SU(n)$ or $U(n)$. Evaluate traces like

$$\mathrm{Tr}\, T^a T^b T^a T^b,$$

where a sum over a and b is implied.

The Fierz identity reads for $SU(N)$:

$$T^a_{ij} T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right).$$

Proof: T^a and the unit matrix form a basis of the $N \times N$ hermitian matrices, therefore any hermitian matrix A can be written as

$$A = c_0 1 + c_a T^a.$$

The constants c_0 and c_a are determined using the normalization condition and the fact that the T^a are traceless. We first take the trace on both sides:

$$\mathrm{Tr}(A) = c_0 \,\mathrm{Tr}\, 1 + c_a \,\mathrm{Tr}\, T^a = c_0 N,$$

therefore

$$c_0 = \frac{1}{N}\mathrm{Tr}(A).$$

Now we multiply first both sides with T^b and take then the trace:

$$\mathrm{Tr}\left(AT^b\right) = c_0 \,\mathrm{Tr}\, T^b + c_a \,\mathrm{Tr}\, T^a T^b = c_a \frac{1}{2}\delta^{ab},$$

therefore

$$c_a = 2\mathrm{Tr}(T^a A).$$

Putting both results together we obtain

$$A = \frac{1}{N}\mathrm{Tr}(A)\, 1 + 2\mathrm{Tr}(AT^a)\, T^a$$

Let us now write this equation in components

$$\begin{aligned} A_{ij} &= \frac{1}{N}\mathrm{Tr}(A)\, 1_{ij} + 2\mathrm{Tr}(AT^a)\, T^a_{ij}, \\ A_{ij} &= \frac{1}{N}A_{ll} 1_{ij} + 2A_{lk} T^a_{kl} T^a_{ij}, \end{aligned}$$

Therefore

$$A_{lk}\left(2T^a_{ij}T^a_{kl} + \frac{1}{N}\delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk}\right) = 0.$$

This has to hold for an arbitrary A, therefore the Fierz identity follows. Useful formulae involving traces:

$$\begin{aligned} \mathrm{Tr}(T^a X)\,\mathrm{Tr}(T^a Y) &= \frac{1}{2}\left[\mathrm{Tr}(XY) - \frac{1}{N}\mathrm{Tr}(X)\,\mathrm{Tr}(Y)\right], \\ \mathrm{Tr}(T^a X T^a Y) &= \frac{1}{2}\left[\mathrm{Tr}(X)\,\mathrm{Tr}(Y) - \frac{1}{N}\mathrm{Tr}(XY)\right]. \end{aligned}$$

The Fierz identity for $U(N)$ reads

$$T^a_{ij} T^a_{kl} \;=\; \frac{1}{2}\delta_{il}\delta_{jk}.$$

From

$$[T^a, T^b] \;=\; i f^{abc} T^c$$

one derives by multiplying with T^d and taking the trace:

$$i f^{abc} \;=\; 2\left[\mathrm{Tr}\left(T^a T^b T^c\right) - \mathrm{Tr}\left(T^b T^a T^c\right)\right]$$

This yields an expression of the structure constants in terms of the matrices of the fundamental representation. We can now calculate for the group $SU(N)$ the fundamental and the adjoint Casimirs:

$$\begin{aligned}
(T^a T^a)_{ij} &= C_F \delta_{ij} = \frac{N^2-1}{2N}\delta_{ij}, \\
f^{abc} f^{dbc} &= C_A \delta^{ad} = N \delta^{ad}.
\end{aligned}$$

3 Representation theory

3.1 Group actions

An action of a group G on a set X is a correspondence that associates to each element $g \in G$ a map $\phi_g : X \to X$ in such a way that

$$\phi_{g_1 g_2} = \phi_{g_1}\phi_{g_2},$$
$$\phi_e \text{ is the identity map on } X,$$

where e denotes the neutral element of the group. Instead of $\phi_g(x)$ one often writes gx.

A group action of G on X gives rise to a natural equivalence relation on X: $x_1 \in X$ and $x_2 \in X$ are equivalent, if they can be obtained from one another by the action of some group element $g \in G$. The equivalence class of a point $x \in X$ is called the **orbit** of x.

G is said to act **effectively** on X, if the homomorphism from G into the group of transformations of X is injective.

G is said to act **transitively** on X, if there is only one orbit. A set X where a group G acts transitively is called a homogeneous space. Every orbit of a (not necessarily transitive) group action is a homogeneous space.

The stabilizer (or the isotropy subgroup or the little group) H_x of a point $x \in X$ is the subgroup of G that leave x fixed, e.g. $h \in H_x$ if $hx = x$. When H_x is the trivial subgroup for all $x \in X$, we say that the action of G on X is **free**.

If G acts on X and on Y, then a map $\psi : X \to Y$ is said to be **G-equivariant** if $\psi \circ g = g \circ \psi$ for all $g \in G$.

3.2 Representations

Let V be a finite-dimensional vector space and $GL(V)$ the group of automorphisms of V. Typically $V = \mathbb{R}^n$ or $V = \mathbb{C}^n$ and $GL(V) = GL(n,\mathbb{R})$ or $GL(V) = GL(n,\mathbb{C})$.

Definition: A representation of a goup G is a homomorphism ρ from G to $GL(V)$

$$g \rightarrow \rho(g).$$

The composition in $GL(V)$ is given by matrix multiplication. Since ρ is a homomorphism we have

$$\rho(g_1 g_2) = \rho(g_1)\rho(g_2).$$

This implies

$$\begin{aligned}\rho(e) &= 1,\\ \rho\left(g^{-1}\right) &= [\rho(g)]^{-1}.\end{aligned}$$

The trivial representation:

$$\rho(g) = 1, \quad \forall g.$$

Remark: In general more than one group element can be mapped on the identity. If the mapping $\rho : G \to GL(V)$ is one-to-one, i.e.

$$\rho(g) = \rho(g') \quad \Rightarrow \quad g = g'$$

then the representation is called **faithful**.

Strictly speaking a representation is a set of (non-singular) matrices, e.g. a sub-set of $GL(V)$. Very often we will also speak about the vector space V, on which these matrices act, as a representation of G.

In this sense a **sub-representation** of a representation V is a vector sub-space W of V, which is invariant under G:

$$\rho(g)w \in W \ \ \forall g \in G \ \text{ and } w \in W.$$

A representation V is called **irreducible** if there is no proper non-zero invariant sub-space W of V. (This excludes the trivial invariant sub-spaces $W = \{0\}$ and $W = V$.)

If V_1 and V_2 are representations of G, the **direct sum** $V_1 \oplus V_2$ and the **tensor product** $V_1 \otimes V_2$ are again representations:

$$\begin{aligned}g\left(v_1 \oplus v_2\right) &= \left(gv_1\right) \oplus \left(gv_2\right),\\ g\left(v_1 \otimes v_2\right) &= \left(gv_1\right) \otimes \left(gv_2\right),\end{aligned}$$

Two representations ρ_1 and ρ_2 of the same dimension are called **equivalent**, if there exists a non-singular matrix S such that

$$\rho_1(g) = S\rho_2(g)S^{-1}, \quad \forall g \in G.$$

For finite groups and Lie groups it can be shown that any representation is equivalent to a unitary representation.

The goal of representation theory: Classify and study all representations of a group G up to equivalence. This will be done by decomposing an arbitrary representation into direct sums of irreducible representations.

3.3 Schur's lemmas

Lemma 1: Any matrix M which commutes with all the matrices $\rho(g)$ of an irreducible representation of a group G must be a multiple of the unit matrix:

$$M = c\mathbf{1}.$$

Proof: We have

$$\rho(g)M = M\rho(g) \quad \forall g \in G.$$

If $\rho(g)$ is of dimension n, then M must be square of dimension n. Let us assume that $\rho(g)$ is unitary. Then

$$M^\dagger \rho(g)^\dagger = \rho(g)^\dagger M^\dagger.$$

Multiply by $\rho(g)$ from left and right:

$$\rho(g)M^\dagger = M^\dagger \rho(g).$$

Therefore also $M^\dagger$ commutes with all $\rho(g)$, and so do the hermitian matrices

$$\begin{aligned} H_1 &= M + M^\dagger, \\ H_2 &= i\left(M - M^\dagger\right). \end{aligned}$$

Any hermitian matrix may be diagonalised by a unitary transformation:

$$D = U^{-1} H U.$$

If we define now

$$\rho'(g) = U^{-1}\rho(g)U,$$

we have

$$\rho'(g)D = D\rho'(g).$$

Let $D = \text{diag}(\lambda_1, ..., \lambda_n)$ and consider now the i, j element of this matrix equation:

$$\begin{aligned} \left[\rho'(g)\right]_{ij} \lambda_j &= \lambda_i \left[\rho'(g)\right]_{ij}, \\ \left(\lambda_i - \lambda_j\right)\left[\rho'(g)\right]_{ij} &= 0. \end{aligned}$$

Suppose that a certain eigenvalue λ of D occurs k times and that, by a suitable ordering the first k positions of D are occupied by λ. Then

$$\lambda_1 = ... = \lambda_k \neq \lambda_l, \;\; k+1 \leq l \leq n.$$

This implies that

$$\left[\rho'(g)\right]_{ij} = 0 \qquad \begin{array}{l} \text{for } 1 \le i \le k, \quad k+1 \le j \le n, \\ \text{or } 1 \le j \le k, \quad k+1 \le i \le n. \end{array}$$

Hence $\rho'(g)$ is of the form

$$\begin{pmatrix} \dots & 0 \\ 0 & \dots \end{pmatrix}$$

and is thus reducible, contrary to the initial assumption. Thus if and only if all the eigenvalues of D are the same $\rho'(g)$ will be irreducible. In other words, D and hence M must be a multiple of the unit matrix.

Lemma 2: If $\rho_1(g)$ and $\rho_2(g)$ are two irreducible representations of a group G of dimensions n_1 and n_2 respectively and if a rectangular matrix M of dimension $n_1 \times n_2$ exists sucht that

$$\rho_1(g)M = M\rho_2(g), \quad \forall g \in G$$

then either
(a) $M = 0$ or
(b) $n_1 = n_2$ and $\det M \neq 0$, in which case $\rho_1(g)$ and $\rho_2(g)$ are equivalent.

Proof: Let us assume without loss of generality that $\rho_1(g)$ and $\rho_2(g)$ are unitary representations.

$$\begin{aligned} M^\dagger \rho_1(g)^\dagger &= \rho_2(g)^\dagger M^\dagger, \\ M^\dagger \rho_1(g^{-1}) &= \rho_2(g^{-1}) M^\dagger. \end{aligned}$$

Multiply by M from the right:

$$M^\dagger \rho_1(g^{-1}) M = \rho_2(g^{-1}) M^\dagger M.$$

By assumption $\rho_1(g^{-1})M = M\rho_2(g^{-1})$ and therefore

$$M^\dagger M \rho_2(g^{-1}) = \rho_2(g^{-1}) M^\dagger M.$$

By lemma 1 we conclude

$$M^\dagger M = \lambda \mathbf{1}.$$

Consider the case $n_1 = n_2 = n$:

$$\det M^\dagger M = \det M^\dagger \det M = \lambda^n.$$

If $\lambda \neq 0$ then $\det M \neq 0$ and therefore M^{-1} exists. From $\rho_1(g)M = M\rho_2(g)$ it follows that

$$\rho_1(g) = M\rho_2(g)M^{-1}$$

and $\rho_1(g)$ and $\rho_2(g)$ are equivalent.

If on the other hand $\lambda = 0$ we have

$$\begin{aligned} \sum_k M^\dagger_{ik} M_{ki} &= 0, \\ \sum_k |M_{ik}|^2 &= 0. \end{aligned}$$

This is only possible for $M_{ik} = 0$ and hence

$$M = 0.$$

To complete the proof we consider the case $n_1 \neq n_2$. Let us assume $n_1 < n_2$. Construct M' from M by adding $n_2 - n_1$ rows of zeros:

$$M' = \begin{pmatrix} M \\ 0 \end{pmatrix}$$

$$M'^\dagger = \begin{pmatrix} M^\dagger & 0 \end{pmatrix}$$

We have

$$M'^\dagger M' = M^\dagger M$$

and thus

$$\det M^\dagger M = \det M'^\dagger M' = \det M'^\dagger \det M' = 0.$$

Hence $\lambda = 0$ and $M^\dagger M = 0$. It follows $M = 0$ as before.

Application: Orthogonality theorem for finite groups. Let G be a finite group and let ρ_1 and ρ_2 be representations of dimension n_1 and n_2. Then

$$\sum_{g \in G} \rho_1(g)_{ij} \rho_2(g^{-1})_{kl} = \begin{cases} 0 & \rho_1 \text{ and } \rho_2 \text{ are inequivalent,} \\ \frac{|G|}{n_1} \delta_{il} \delta_{kj} & \rho_1 \text{ and } \rho_2 \text{ are identical,} \\ \dots & \rho_1 \text{ and } \rho_2 \text{ are equivalent, but not identical.} \end{cases}$$

Proof: Assume that ρ_1 and ρ_2 are inequivalent. Consider

$$M = \frac{1}{|G|} \sum_{g \in G} \rho_1(g) X \rho_2(g^{-1}),$$

where X is an arbitrary $n_1 \times n_2$ matrix. Then

$$\begin{aligned} \rho(g')M &= \rho(g') \frac{1}{|G|} \sum_{g \in G} \rho_1(g) X \rho_2(g^{-1}) = \frac{1}{|G|} \sum_{g \in G} \rho_1(g'g) X \rho_2(g^{-1}) \\ &= \frac{1}{|G|} \sum_{g \in G} \rho_1(g) X \rho_2(g^{-1}g') = \frac{1}{|G|} \sum_{g \in G} \rho_1(g) X \rho_2(g^{-1}) \rho(g') = M\rho(g'). \end{aligned}$$

By Schur's second lemma we have $M = 0$, therefore

$$\frac{1}{|G|}\sum_{g\in G}\rho_1(g)_{ij'}X_{j'k'}\rho_2(g^{-1})_{k'l} = 0.$$

Since X was arbitrary we can take $X = \delta_{jj'}\delta kk'$ and we have

$$\sum_{g\in G}\rho_1(g)_{ij}\rho_2(g^{-1})_{kl} = 0.$$

Now consider the case where ρ_1 and ρ_2 are identical: $\rho_1 = \rho_2 = \rho$. Take again

$$M = \frac{1}{|G|}\sum_{g\in G}\rho(g)X\rho(g^{-1}).$$

One shows again

$$\rho(g)M = M\rho(g).$$

Therefore by Schur's first lemma

$$\frac{1}{|G|}\sum_{g\in G}\rho(g)_{ij'}X_{j'k'}\rho(g^{-1})_{k'l} = c\delta_{il}.$$

Again take $X = \delta_{jj'}\delta kk'$:

$$\frac{1}{|G|}\sum_{g\in G}\rho(g)_{ij}\rho(g^{-1})_{kl} = c\delta_{il}.$$

To find c take the trace on both sides:

$$\delta_{kj} = cn_1,$$

and therefore

$$\sum_{g\in G}\rho(g)_{ij}\rho(g^{-1})_{kl} = \frac{|G|}{n_1}\delta_{kj}\delta_{il}.$$

Another consequence of Schur's first lemma: All irreducible representation of an Abelian group are one-dimensional.

3.4 Representation theory for finite groups

A finite group G admits only finitely many irreducible representations V_i up to isomorphism.

Example: Consider the symmetric group S_3, the permutation group of three elements, which is the simplest non-abelian group. This group has two one-dimensional representations: The trivial one I and the alternating representation A defined by

$$gv = \text{sign}(g)v.$$

There is a natural representation, in which S_3 acts on $\mathbb{C}^3$ by

$$g \cdot (z_1, z_2, z_3) = \left(z_{g^{-1}(1)}, z_{g^{-1}(2)}, z_{g^{-1}(3)}\right)$$

This representation is reducible: The line spanned by the sum

$$e_1 + e_2 + e_3$$

is an invariant sub-space. The complementary sub-space

$$V = \{(z_1, z_2, z_3) | z_1 + z_2 + z_3 = 0\}$$

defines an irreducible representation. This representation is called the standard representation.
It can be shown that any representation of S_3 can be decomposed into these three irreducible representations

$$W = I^{\oplus n_1} \oplus A^{\oplus n_2} \oplus V^{\oplus n_3}.$$

3.4.1 Characters

Definition:If V is a representation of G, its character χ_V is the complex-valued function on the group defined by

$$\chi_V(g) = \text{Tr}(\rho(g)).$$

In particular we have

$$\chi_V\left(hgh^{-1}\right) = \chi_V(g).$$

Let V and W be representation of G. Then

$$\begin{aligned}
\chi_{V\oplus W} &= \chi_V + \chi_W, \\
\chi_{V\otimes W} &= \chi_V \cdot \chi_W, \\
\chi_{V^*} &= (\chi_V)^*, \\
\chi_{\wedge^2 V}(g) &= \frac{1}{2}\left[\chi_V(g)^2 - \chi_V(g^2)\right], \\
\chi_{\text{Sym}^2 V}(g) &= \frac{1}{2}\left[\chi_V(g)^2 + \chi_V(g^2)\right]
\end{aligned}$$

Orthogonality theorem for characters: For finite groups we had the orthogonality theorem. If we consider unitary representations and if we make the agreement that if two representations are equivalent, we take them to be identical, the orthogonality theorem can be written as

$$\sum_{g\in G} \rho_a(g)_{ij} \rho_b(g)^*_{lk} = \frac{|G|}{n_1} \delta_{il} \delta_{kj} \delta_{ab}$$

Now we set $i = j$ and sum, and we set $l = k$ and sum:

$$\sum_{g\in G} \chi_a(g)\chi_b(g)^* = |G|\,\delta_{ab}$$

Since the character is a class function we can write

$$\sum_{g\in G} = \sum_{\text{classes }\kappa} n_\kappa,$$

where n_κ denotes the number of elements in the class C_κ. Therefore

$$\sum_\kappa n_\kappa \chi_a(C_\kappa)\chi_b(C_\kappa)^* = |G|\,\delta_{ab}$$

Character table:

	n_1C_1	n_2C_2	n_2C_2	...
ρ_1	$\chi_1(C_1)$	$\chi_1(C_2)$	$\chi_1(C_3)$	...
ρ_2	$\chi_2(C_1)$	$\chi_2(C_2)$	$\chi_2(C_3)$	...
ρ_3	$\chi_3(C_1)$	$\chi_3(C_2)$	$\chi_3(C_3)$	...
...	...	...	...	...

If we now define

$$\zeta^a_\kappa = \sqrt{\frac{n_\kappa}{|G|}}\chi_a(C_\kappa)$$

we have

$$\sum_\kappa \zeta^a_\kappa \zeta^{b*}_\kappa = \delta^{ab}.$$

The number of orthogonal vectors corresponds to the number of inequivalent representations. The dimension of the space is given by the number of classes. Therefore the number of inequivalent representations is smaller or equal to the number of classes. In fact equality holds.

Criteria for reducibility: Assume that

$$\rho(g) = \bigoplus_\alpha a_\alpha \rho_\alpha(g)$$

Then

$$\chi(g) = \sum_\alpha a_\alpha \chi_\alpha(g).$$

Conisder now

$$\frac{1}{|G|}\sum_g \chi(g)\chi(g)^* = \sum_\alpha |a_\alpha|^2 \quad \begin{cases} = 1 & \rho \text{ irreducible} \\ > 1 & \rho \text{ reducible} \end{cases}$$

3.5 Representation theory for Lie groups

3.5.1 Irreducible representation of $SU(2)$ and $SO(3)$

The groups $SU(2)$ and $SO(3)$ have the same Lie algebra:

$$[I_a, I_b] = i\varepsilon_{abc} I_c.$$

For $SU(2)$ we can take the I^a's proportional to the Pauli matrices

$$I_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad I_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad I_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This defines a representation of $SU(2)$ which is called the fundamental representation. (It is not a representation of $SO(3)$.)
Quite generally the structure constants provide a representation known as the adjoint or vector representation:

$$(M_b)_{ac} = i f_{abc}.$$

For $SU(2)$ and $SO(3)$:

$$M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The dimension of the adjoint representation equals the dimension of the parameter space of the group and the numbers of generators.

Let us now discus more systematically all irreducible representations.
Definition: A **Casimir operator** is an operator, which commutes with all the generators of the group.
Example: For $SU(2)$ and $SO(3)$

$$I^2 = I_1^2 + I_2^2 + I_3^2$$

is a Casimir operator:

$$\left[I^2, I_a\right] = 0.$$

Definition: The **rank of a Lie algebra** is the number of simultaneously diagonalisable generators.
Example 1: $SU(2)$ has rank one, the convention is to take I_3 diagonal.
Example 2: $SU(3)$ has rank two, in the Gell-Mann representation T_3 and T_8 are diagonal.

Theorem: The number of independent Casimir operators is equal to the rank of the Lie algebra. The proof can be found in many textbooks.

Therefore $SU(2)$ has only one Casimir operator.

The eigenvalues of the Casimir operators may be used to label the irreducible representations. The eigenvalues of the diagonal generators can be used to label the basis vectors within a given irreducible representation.

Example $SU(2)$:

$$\begin{aligned} I^2 |\lambda, m\rangle &= \lambda |\lambda, m\rangle, \\ I_3 |\lambda, m\rangle &= m |\lambda, m\rangle. \end{aligned}$$

Consider

$$\left(I_1^2 + I_2^2\right) |\lambda, m\rangle = \left(I^2 - I_3^2\right) |\lambda, m\rangle = \left(\lambda - m^2\right) |\lambda, m\rangle.$$

Further

$$\langle\lambda, m| I_1^2 |\lambda, m\rangle = \langle\lambda, m| I_1^\dagger I_1 |\lambda, m\rangle = |I_1 |\lambda, m\rangle|^2 \geq 0.$$

A similar consideration applies to $\langle\lambda, m| I_2^2 |\lambda, m\rangle$. Therefore

$$\lambda - m^2 \geq 0.$$

For a given λ the possible values of m are bounded:

$$-\sqrt{\lambda} \leq m \leq \sqrt{\lambda}$$

Define

$$I_\pm = \frac{1}{\sqrt{2}} \left(I_1 \pm i I_2\right)$$

$$[I_3, I_\pm] = \pm I_\pm, \qquad \left[I^2, I_\pm\right] = 0.$$

The last relation implies

$$\begin{aligned} \left(I^2 I_\pm - I_\pm I^2\right) |\lambda, m\rangle &= 0, \\ I^2 \left(I_\pm |\lambda, m\rangle\right) &= \lambda \left(I_\pm |\lambda, m\rangle\right). \end{aligned}$$

Therefore the operators $I_\pm$ don't change λ. From the commutation relation with I_3 we obtain

$$\begin{aligned} \left(I_3 I_\pm - I_\pm I_3\right) |\lambda, m\rangle &= \pm I_\pm |\lambda, m\rangle, \\ I_3 \left(I_\pm |\lambda, m\rangle\right) &= (m \pm 1) \left(I_\pm |\lambda, m\rangle\right). \end{aligned}$$

Therefore $I_{\pm} |\lambda, m\rangle$ is proportional to $|\lambda, m \pm 1\rangle$ unless zero. Recall that the values of m are bounded, therefore there is a maximal value m_{max} and a minimal value m_{min}:

$$\begin{aligned} I_{+} |\lambda, m_{max}\rangle &= 0, \\ I_{-} |\lambda, m_{min}\rangle &= 0. \end{aligned}$$

Now

$$\begin{aligned} I^2 = I_1^2 + I_2^2 + I_3^2 &= I_+ I_- + I_3^2 - I_3 \\ &= I_- I_+ + I_3^2 + I_3 \end{aligned}$$

Therefore

$$\begin{aligned} I^2 |\lambda, m_{max}\rangle &= \left(I_- I_+ + I_3^2 + I_3\right) |\lambda, m_{max}\rangle, \\ \lambda |\lambda, m_{max}\rangle &= m_{max} (m_{max} + 1) |\lambda, m_{max}\rangle, \end{aligned}$$

and

$$\lambda = m_{max} (m_{max} + 1).$$

Similar:

$$\begin{aligned} I^2 |\lambda, m_{min}\rangle &= \left(I_+ I_- + I_3^2 - I_3\right) |\lambda, m_{min}\rangle, \\ \lambda |\lambda, m_{min}\rangle &= m_{min} (m_{min} - 1) |\lambda, m_{min}\rangle, \end{aligned}$$

and

$$\lambda = m_{min} (m_{min} - 1).$$

From

$$\begin{aligned} m_{max}^2 + m_{max} &= m_{min}^2 - m_{min}, \\ (m_{max} + m_{min}) \underbrace{(m_{max} - m_{min} + 1)}_{>0} &= 0 \end{aligned}$$

it follows

$$m_{min} = -m_{max}.$$

Since the ladder operators raise or lower m by one unit we must have that m_{max} and m_{min} differ by an integer, therefore

$$2m_{max} = \text{integer.}$$

Let us write $m_{max} = j$. Then $2j$ is an integer and

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, ...$$

$$\lambda = j(j+1)$$

Normalisation:

$$I_\pm |\lambda, m\rangle = A_\pm |\lambda, m\pm 1\rangle$$

With $I_\pm^\dagger = I_\mp$ we have

$$|A_\pm|^2 = \langle\lambda, m| I_\pm^\dagger I_\pm |\lambda, m\rangle = \langle\lambda, m| I_\mp I_\pm |\lambda, m\rangle = \langle\lambda, m| I^2 - I_3 (I_3 \pm 1) |\lambda, m\rangle$$

and therefore

$$|A_\pm|^2 = j(j+1) - m(m\pm 1)$$

Condon-Shortley convention:

$$A_\pm = \sqrt{j(j+1) - m(m\pm 1)}.$$

3.5.2 The Cartan basis

Definition: Suppose a Lie algebra A has a sub-algebra B such that the commutator of any element of A (T^a say) with any element of B (T^b say) always lies in B, then B is said to be an **ideal** of A:

$$\left[T^a, T^b\right] \in B.$$

Every Lie algebra has two trivial ideals: A and $\{0\}$.

A Lie algebra is called **simple** if it is non-Abelian and has no non-trivial ideals.

A Lie algebra is called **semi-simple** if it has no non-trivial Abelian ideals.

A Lie algebra is called **reductive** if it is the sum of a semi-simple and an abelian Lie algebra.

A simple Lie algebra is also semi-simple and a semi-simple Lie algebra is also reductive.

Examples: The Lie algebras

$$su(n), so(n), sp(n)$$

are simple.
Semi-simple Lie algebras are sums of simple Lie algebras:

$$su(n_1) \oplus su(n_2).$$

Reductive Lie algebras may have in addition an abelian part:

$$u(1)\oplus su(2)\oplus su(3).$$

From Schur's lemma we know that abelian Lie groups have only one-dimensional irreducible representations. Therefore let us focus on Lie groups corresponding to semi-simple Lie algebras. A Lie group, which has a semi-simple Lie algebra, is for obvious reasons called semi-simple. We first would like to have a criterion to decide, whether a Lie algebra is semi-simple or not: If

$$\left[T^a,T^b\right] = if^{abc}T^c,$$

define

$$g^{ab} = f^{acd}f^{bcd}.$$

A criterion due to Cartan say that a Lie algebra is semi-simple if and only if

$$\det g \neq 0.$$

For $SU(n)$ we find

$$g^{ab} = C_A\delta^{ab}.$$

Let us now define the Cartan standard form of a Lie algebra. As example suppose we have

$$\left[T^1,T^2\right]=0,\quad \left[T^1,T^3\right]\neq 0,\quad \left[T^2,T^3\right]\neq 0.$$

If we now make a change of basis

$$T^{1'}=T^1+T^3,\quad T^{2'}=T^2,\quad T^{3'}=T^3,$$

none of the new commutators vanishes. More generally let us assume that

$$\begin{aligned} A &= i\sum_{i=1}^{n} c_aT^a, \\ X &= i\sum_{i=1}^{n} x_aT^a, \end{aligned}$$

such that

$$[A,X] = i\rho X.$$

ρ is called a **root** of the Lie algebra. We then have

$$\begin{aligned} [A,X] &= -ic_ax_bf^{abc}T^c = -\rho x_cT^c, \\ \left(c_ax_bif^{abc}-\rho x_c\right) &= 0, \\ \left(c_aif^{abc}-\rho\delta^{bc}\right)x_b &= 0. \end{aligned}$$

For a non-trivial solution we must have

$$\det\left(c_a i f^{abc} - \rho\delta^{bc}\right) = 0.$$

In general the secular equation will give a n-th order polynomial is ρ. Solving for ρ one obtains n roots. One root may occur more than once. The degree of degeneracy is called the multiplicity of the root.

Theorem (Cartan): If A is chosen sucht that the secular equation has the maximum number of distince roots, then only the root $\rho = 0$ is degenerate. Further if r is the multiplicity of that root, there exist r linearly independent eigenvectors H_i, which mutually commute

$$\left[H_i, H_j\right] = 0, \quad i, j = 1, ..., r.$$

r is the rank of the Lie algebra.

Notation: Latin indices for $1, ..., r$, e.g. H_i and greek indices for the remaining $(n-r)$ generators E_α $(\alpha = 1, ..., n-r)$.

Example $SU(2)$:

$$\left[I^a, I^b\right] = i\varepsilon^{abc} I^c.$$

Take $A = iI^3$:

$$\left[iI^3, X\right] = i\rho X.$$

Secular equation:

$$\begin{aligned}
\det\left(i\varepsilon^{3bc} - \rho\delta^{bc}\right) &= 0, \\
\begin{vmatrix} -\rho & i & 0 \\ -i & -\rho & 0 \\ 0 & 0 & -\rho \end{vmatrix} &= 0, \\
-\rho^3 + \rho &= 0, \\
\rho(\rho - 1) &= 0.
\end{aligned}$$

Therefore the roots are $0, \pm 1$. We have

$$\begin{aligned}
\rho = 0 &\qquad \left[I^3, X\right] = 0 \quad \Rightarrow X = I^3 = H_1, \\
\rho = 1 &\qquad \left[I^3, X\right] = X \quad \Rightarrow X = \frac{1}{\sqrt{2}}\left(I^1 + iI^2\right) = E_1, \\
\rho = -1 &\qquad \left[I^3, X\right] = -X \quad \Rightarrow X = \frac{1}{\sqrt{2}}\left(I^1 - iI^2\right) = E_2.
\end{aligned}$$

Theorem: For any compact semi-simple Lie group, non-zero roots occur in pairs of opposite sign and are denoted E_α and $E_{-\alpha}$ ($\alpha = 1, ..., (n-r)/2$).

We thus have the Cartan standard form:

$$\begin{aligned} [H_i, H_j] &= 0, \\ [H_i, E_\alpha] &= \rho(\alpha, i) E_\alpha. \end{aligned}$$

As a short-hand notation the last equation is also often written as

$$[H_i, E_\alpha] = \alpha_i E_\alpha.$$

The standard normalisation for the Cartan basis is

$$\sum_{\alpha=1}^{(n-r)/2} \rho(\alpha, i) \rho(\alpha, j) = \delta_{ij}.$$

Cartan standard form of $SU(2)$:

$$H_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Cartan standard form of $SU(3)$:

$$H_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \frac{1}{3\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

$$E_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_{-1} = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{-2} = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad E_{-3} = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The r numbers α_i, $i = 1, ..., r$ can be regarded as the components of a root vector $\vec{\alpha} = (\alpha_1, ..., \alpha_r)$ of dimension r.

Theorem: If $\vec{\alpha}$ is a root vector, so is $-\vec{\alpha}$, (since roots always occur in pairs of opposite sign).

Theorem: If $\vec{\alpha}$ and $\vec{\beta}$ are root vectors then

$$\frac{2\vec{\alpha}\cdot\vec{\beta}}{|\alpha|^2} \quad \text{and} \quad \frac{2\vec{\alpha}\cdot\vec{\beta}}{|\beta|^2}$$

are integers. Suppose these integers are p and q. Then

$$\frac{\left(\vec{\alpha}\cdot\vec{\beta}\right)^2}{|\alpha|^2\,|\beta|^2} = \frac{pq}{4} = \cos^2\theta \leq 1.$$

Therefore

$$pq \leq 4.$$

It follows that

$$\cos^2\theta = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1.$$

Case $\theta = 0^\circ$: This is the trivial case $\vec{\alpha} = \vec{\beta}$.
Case $\theta = 30^\circ$: We have $pq = 3$ and $p = 1, q = 3$ or $p = 3, q = 1$. Let us first discuss $p = 1, q = 3$. This means

$$\frac{2\vec{\alpha}\cdot\vec{\beta}}{|\alpha|^2} = 1, \qquad \frac{2\vec{\alpha}\cdot\vec{\beta}}{|\beta|^2} = 3.$$

Therefore

$$\frac{|\alpha|^2}{|\beta|^2} = 3.$$

The case $p = 3, q = 1$ is similar and in summary we obtain

$$\frac{|\alpha|^2}{|\beta|^2} = 3 \text{ or } \frac{1}{3}.$$

Case $\theta = 45^\circ$: We have $pq = 2$ and $p = 1, q = 2$ or $p = 2, q = 1$. It follows

$$\frac{|\alpha|^2}{|\beta|^2} = 2 \text{ or } \frac{1}{2}.$$

Case $\theta = 60^\circ$: We have $pq = 1$ and $p = 1, q = 1$. It follows

$$\frac{|\alpha|^2}{|\beta|^2} = 1.$$

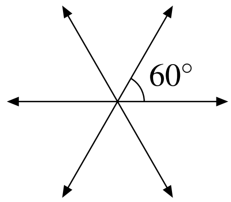
Case $\theta = 90^\circ$: In this case $p = 0$ and $q = 0$. This leaves the ratio $|\alpha|^2/|\beta|^2$ undetermined.

The cases $\theta = 120^\circ$, $\theta = 135^\circ$ and $\theta = 150^\circ$ are analogous to the ones discussed above.

If $\vec{\alpha}$ and $\vec{\beta}$ are root vectors so is

$$\vec{\gamma} = \vec{\beta} - \frac{2\vec{\alpha}\cdot\vec{\beta}}{\alpha^2}\vec{\alpha}$$

Example: The root diagram of $SU(3)$:



3.5.3 Weights

Let us first recall some basic facts: The rank of a Lie algebra is the number of simultaneously diagonalizable generators. In the following we will denote the rank of a Lie algebra by r.

Theorem : The rank of a Lie algebra is equal to the number of independent Casimir operators. (A Casimir operator is an operator, which commutes with all the generators.)

For a Lie algebra of rank r we therefore have r Casimir operators and r simultaneously diagonalizable generators H_i.

The eigenvalues of the Casimir operators may be used to label the irreducible representations. The eigenvalues of the diagonal generators H_i may be used to label the states within a given irreducible representation.

Let $\vec{\lambda}$ be a shorthand notation for $\vec{\lambda} = (\lambda_1, ..., \lambda_r)$, a set of eigenvalues of Casimir operators and let $\vec{m}$ be a shorthand notation for $\vec{m} = (m_1, ..., m_r)$, a set of eigenvalues of the diagonal generators:

$$H_i |\vec{\lambda}, \vec{m}\rangle = m_i |\vec{\lambda}, \vec{m}\rangle$$

The vector $\vec{m}$ is called the weight vector.

Example $SU(3)$: Let us consider the fundamental representation. The vector space is spanned by the three vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

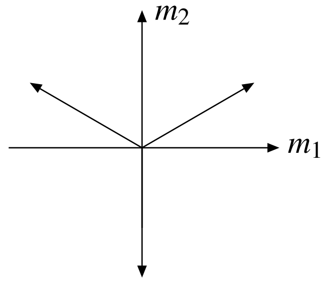
We have

$$\begin{aligned} (H_1, H_2)\, e_1 &= \left(\frac{1}{\sqrt{6}}, \frac{1}{3\sqrt{2}} \right) e_1, \\ (H_1, H_2)\, e_2 &= \left(-\frac{1}{\sqrt{6}}, \frac{1}{3\sqrt{2}} \right) e_2, \\ (H_1, H_2)\, e_3 &= \left(0, -\frac{2}{3\sqrt{2}} \right) e_3. \end{aligned}$$

This gives the weight vectors

$$\vec{m}_1 = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{3\sqrt{2}} \end{pmatrix}, \quad \vec{m}_2 = \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{3\sqrt{2}} \end{pmatrix}, \quad \vec{m}_3 = \begin{pmatrix} 0 \\ -\frac{\sqrt{2}}{3} \end{pmatrix}.$$

and the weight diagram



Consider now the complex conjugate representation of the fundamental representation: If

$$\rho = \exp(i\theta_a T^a)$$

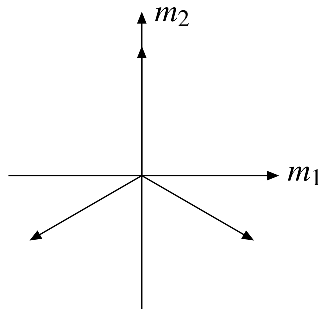
is a representation, then also

$$\rho^* = \exp(-i\theta_a T^{a*}) = \exp\left(i\theta_a T^{a\prime}\right)$$

is a representation and we have

$$T^{a\prime} = -T^{a*}.$$

It follows that the weights of the complex conjugate representation are negatives of those of the fundamental representation:



Note that in general the complex conjugate representation ρ^* is inequivalent to ρ. This is in contrast to $SU(2)$, where one can find a S, such that

$$SI^aS^{-1} = -I^{a*}, \quad S\rho S^{-1} = \rho^*.$$

Let us now look at the weights from a more general perspective: The number of different eigenstates with the same weight is called the multiplicity of the weight. A weight is said to be simple if the multiplicity is 1.

For Lie algebras with $r \geq 2$, weights are not necessarily simple.

Theorem : Given a weight $\vec{m}$ and a root vector $\vec{\alpha}$ then

$$\frac{2\vec{\alpha}\cdot\vec{m}}{\alpha^2}$$

is an integer and

$$\vec{m}' = \vec{m} - \frac{2\vec{\alpha}\cdot\vec{m}}{\alpha^2}\vec{\alpha}$$

is also a weight vector. $\vec{m}$ and $\vec{m}'$ are called equivalent weights.
(Recall: In the $SU(2)$ case the weight vectors were one-dimensional. Within one irreducible representation all weights could be obtained from m_{max} by applying the lowering operator I_-. The action of I_- corresponds to a shift in the weight proportional to a root vector. For $SU(2)$ all weights within an irreducible representation are equivalent.)

Ordering of weights: The convention for $SU(n)$ is the following: $\vec{m}$ is said to be higher than $\vec{m}'$ if the r^{th} component of $(\vec{m}-\vec{m}')$ is positive (if zero look at the $(r-1)^{\text{th}}$ component).

The highest weight of a set of equivalent weights is said to be **dominant**.
(In the case of an irreducible representation of $SU(2)$ the dominant weight is the one with m_{max}.)

Theorem: For any compact semi-simple Lie algebra there exists for any irreducible representation a highest weight. Furthermore this highest weight is also simple. All other weights of the irreducible representation are equivalent to this one. Therefore the highest weights is also dominant.
(Recall: In the $SU(2)$ case we first showed that the values of m are bounded, and then obtained all other states in the irreducible representation by applying the lowering operator to the state with m_{max}.)

Theorem: For every simple Lie algebra of rank r there are r dominant weights $\vec{M}^{(i)}$, called fundamental dominant weights, such that any other dominant weight $\vec{M}$ is a linear combination of the $\vec{M}^{(i)}$

$$\vec{M} = \sum_{i=1}^{r} n_i \vec{M}^{(i)}$$

where the n_i are non-negative integers.

Note that there exists r fundamental irreducible representations, which have the r $\vec{M}^{(i)}$ as their highest weight. We can label the irreducible representations by $(n_1, n_2, ..., n_r)$ instead of the eigenvalues of the Casimirs.

3.6 Tensor methods

We have already seen how to construct new representation out of given ones through the operations of the direct sum and the tensor product: If V_1 and V_2 are representations of G, the direct sum $V_1 \oplus V_2$ and the tensor product $V_1 \otimes V_2$ are again representations:

$$\begin{aligned} g\left(v_1 \oplus v_2\right) &= \left(gv_1\right) \oplus \left(gv_2\right), \\ g\left(v_1 \otimes v_2\right) &= \left(gv_1\right) \otimes \left(gv_2\right), \end{aligned}$$

We now turn to the question how to construct new irreducible representations out of given irreducible ones. If V_1 and V_2 are irreducible representations, the direct sum is $V_1 \oplus V_2$ is reducible and decomposes into the irreducible representations V_1 and V_2. Nothing new here. More interesting is the tensor product, which we will study in the following.

3.6.1 Clebsch-Gordan series

To motivate the discussion of tensor methods we start again from the $SU(2)$ example and its relation to the spin of a physical system. Suppose we have to independent spin operators $\vec{J}_1$ and $\vec{J}_2$, describing the spin of particle 1 and 2, respectively.

$$\left[J_{1\,i}, J_{2\,j}\right] = 0 \quad \forall i, j$$

Let us now define the total spin as

$$\begin{aligned} \vec{J} &= \vec{J}_1 + \vec{J}_2, \\ J_z &= J_{1\,z} + J_{2\,z}. \end{aligned}$$

We use the following notation:

$$\begin{aligned} &|j_1, m_1\rangle \qquad \text{eigenstate of } J_1^2 \text{ and } J_{1\,z} \\ &|j_2, m_2\rangle \qquad \text{eigenstate of } J_2^2 \text{ and } J_{2\,z} \end{aligned}$$

We define

$$|j_1, j_2, m_1, m_2\rangle = |j_1, m_1\rangle \otimes |j_2, m_2\rangle.$$

The set

$$\{|j_1, j_2, m_1, m_2\rangle\}$$

are eigenvectors of

$$\left\{J_1^2, J_2^2, J_{1\,z}, J_{2\,z}\right\}$$

and is referred to as the uncoupled basis. In general these states are not eigenstates of J^2 and the basis is reducible. This can be seen easily:

$$J^2 = \left(\vec{J}_1 + \vec{J}_2\right)\left(\vec{J}_1 + \vec{J}_2\right) = J_1^2 + J_2^2 + 2\vec{J}_1\vec{J}_2,$$

and $2\vec{J}_1\vec{J}_2$ fails to commute with $J_{1\,z}$ and $J_{2\,z}$. To find a better basis, we look for a set of mutually commuting operators. The set

$$\left\{J^2, J_z, J_1^2, J_2^2\right\}$$

is such a set and an eigenbasis for this set is labelled by

$$\left\{|j,m,j_1,j_2\rangle\right\}.$$

This basis is called the coupled basis and carries and irreducible representation of dimension $2j+1$. Of course we can express each vector in the coupled basis through a linear combination of the uncoupled basis:

$$|j,m,j_1,j_2\rangle \;=\; \sum_{m_1,m_2;m_1+m_2=m} C^{jm}_{j_1 j_2 m_1 m_2} |j_1,j_2,m_1,m_2\rangle$$

The coefficients $C^{jm}_{j_1 j_2 m_1 m_2}$ are called the Clebsch-Gordan coefficients. The Clebsch-Gordan coefficients are tabulated in the particle data group tables.

Example: We take $j_1 = j_2 = 1/2$ and use the short-hand notation

$$\begin{aligned}
|\uparrow\uparrow\rangle &= \left|j_1=\frac{1}{2}, j_2=\frac{1}{2}, m_1=\frac{1}{2}, m_2=\frac{1}{2}\right\rangle, \\
|\uparrow\downarrow\rangle &= \left|j_1=\frac{1}{2}, j_2=\frac{1}{2}, m_1=\frac{1}{2}, m_2=-\frac{1}{2}\right\rangle, \\
|\downarrow\uparrow\rangle &= \left|j_1=\frac{1}{2}, j_2=\frac{1}{2}, m_1=-\frac{1}{2}, m_2=\frac{1}{2}\right\rangle, \\
|\downarrow\downarrow\rangle &= \left|j_1=\frac{1}{2}, j_2=\frac{1}{2}, m_1=-\frac{1}{2}, m_2=-\frac{1}{2}\right\rangle,
\end{aligned}$$

For the coupled basis we have $j \in \{0,1\}$ and we find

$$\begin{aligned}
\left|j=1, m=1, j_1=\frac{1}{2}, j_2=\frac{1}{2}\right\rangle &= |\uparrow\uparrow\rangle, \\
\left|j=1, m=0, j_1=\frac{1}{2}, j_2=\frac{1}{2}\right\rangle &= \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right), \\
\left|j=1, m=-1, j_1=\frac{1}{2}, j_2=\frac{1}{2}\right\rangle &= |\downarrow\downarrow\rangle, \\
\left|j=0, m=0, j_1=\frac{1}{2}, j_2=\frac{1}{2}\right\rangle &= \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right),
\end{aligned}$$

Note that the three states with $j=1$ form an irreducible representation, as does the state with $j=0$. The tensor product of two spin $1/2$ states decomposed therefore as

$$\mathbf{2}\otimes\mathbf{2} \;=\; \mathbf{3}\oplus\mathbf{1},$$

where $\mathbf{n}$ denotes an irreducible representation of dimension n.

3.6.2 The Wigner-Eckart theorem

Let us make a small detour and discuss the Wigner-Eckart theorem. We have already seen that the group $SU(2)$ is generated by the three generators

$$\begin{aligned} J_1 &= \frac{1}{\sqrt{2}}\left(I_1 + iI_2\right), \\ J_0 &= I_3, \\ J_{-1} &= \frac{1}{\sqrt{2}}\left(I_1 - iI_2\right). \end{aligned}$$

$\{J_{-1}, J_0, J_1\}$ define the spherical basis. The generators transform under $SU(2)$ as the adjoint representation. For $SU(2)$ the adjoint representation is the **3** representation. Let us denote the matrix representation of such a transformation by

$$\mathcal{D}^{(j)}_{m'm},$$

where $2j+1$ denotes the dimension of the representation and $m', m = -j, ..., 0, ..., j$. The spherical basis transforms in this notation as

$$\left(J_{q'}\right)' = \mathcal{D}^{(1)}_{q'q} J_q.$$

We can now generalise this construction and define a tensor operator T^k_q of rank k as a set of $(2k+1)$ operators which transform irreducibly under the group as

$$\left(T^k_{q'}\right)' = \mathcal{D}^{(k)}_{q'q} T^k_q.$$

The set $\{J_{-1}, J_0, J_1\}$ is therefore a tensor operator of rank 1. An equivalent definition for a tensor operator is a set of $(2k+1)$ operators satisfying

$$\begin{aligned} \left[I_3, T^k_q\right] &= q T^k_q, \\ \left[I_\pm, T^k_q\right] &= \sqrt{(k \mp q)(k \pm q + 1)}\, T^k_{q\pm 1} = \sqrt{k(k+1) - q(q\pm 1)}\, T^k_q. \end{aligned}$$

We can now state the Wigner-Eckart theorem:

$$\left\langle j'm' \left| T^k_q \right| jm \right\rangle = \frac{1}{\sqrt{2j'+1}} C^{j'm'}_{jkmq} \left\langle j' \left|\left| T^k \right|\right| j \right\rangle.$$

The important point is that the double bar matrix element $\langle j'||T^k||j\rangle$ is independent of m, m' and q. The dependence on m, m' and q is entirely given by the Clebsch-Gordan coefficients $C^{j'm'}_{jkmq}$.

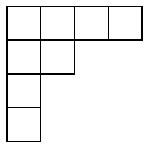
3.6.3 Young diagrams

We have seen that the tensor product of two fundamental representations of $SU(2)$ decomposes as

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1},$$

into a direct sum of irreducible representations. We generalise this now to general irreducible representations of $SU(N)$.

Definition: A Young diagram is a collection of m boxes $\square$ arranged in rows and left-justified. To be a legal Young diagram, the number of boxes in a row must not increase from top to bottom. An example for a Young diagram is



Let us denote the number of boxes in row j by λ_j. Then a Young diagram is a partition of m defined by the numbers $(\lambda_1, \lambda_2, ..., \lambda_n)$ subject to

$$\lambda_1 + \lambda_2 + ... + \lambda_n = m,$$
$$\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n.$$

The example diagram above therefore corresponds to

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (4, 2, 1, 1)$$

The number of rows is denoted by n. For $SU(N)$ we consider only Young diagrams with $n \leq N$.

Let us further define $(n-1)$ numbers p_j by

$$\begin{aligned} p_1 &= \lambda_1 - \lambda_2, \\ p_2 &= \lambda_2 - \lambda_3, \\ &... \\ p_{n-1} &= \lambda_{n-1} - \lambda_{n-2}. \end{aligned}$$

The example above has

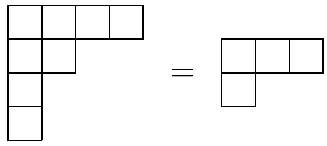
$$(p_1, p_2, p_3) = (2, 1, 0)$$

Correspondence between Young diagrams and irreducible representations: Recall from the last lecture that we could label any irreducible representation of a simple Lie algebra of rank r by either the r eigenvalues of the Casimir operators or by the r numbers $(p_1, ..., p_r)$ appearing when expressing the dominant weight of the representation in terms of the fundamental dominant weights:

$$\vec{M} = \sum_{i=1}^{r} p_i \, \vec{M}^{(i)}$$

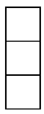
The group $SU(N)$ has rank $N-1$ and we associate to an irreducible representation of $SU(N)$ given through $(p_1, ..., p_{N-1})$ the Young diagram corresponding to $(p_1, ..., p_{N-1})$.

As only differences in the number boxes between succesive rows matter, we are allowed to add any completed column of N boxes from the left. Therefore in $SU(4)$ we have

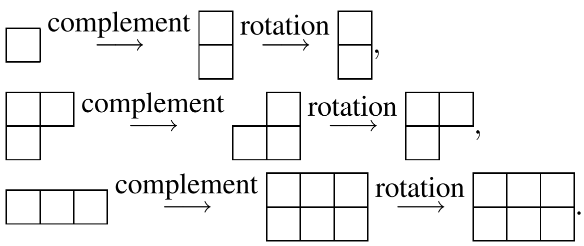


The fundamental representation of $SU(N)$ is always represented by a single box

The trivial (or singlet) representation is alway associated with a column of N boxes. For $SU(3)$:



The complex conjugate representation of a given representation is associated with the conjugate Young diagram. This diagram is obtained by taking the complement with respect to complete columns of N boxes and rotate through 180° to obtain a legal Young diagram.
Examples for $SU(3)$:



The hook rule for the dimensionality of an irreducible representation:

i) Place integers in the boxes, starting with N in the top left box, increase in steps of 1 across rows, decrease in steps of 1 down columns:

N	$N+1$	$N+2$
$N-1$	N	
$N-2$	$N-1$	

ii) Compute the numerator as the product of all integers.

iii) The denominator is given by multiplying all hooks of a Young diagram. A hook is the number of boxes that one passes through on entering the tableau along a row from the right hnad side and leaving down a column.

Some examples for $SU(3)$:

$$\begin{array}{|c|c|}\hline 3&4\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}: \qquad \dim=\frac{2\cdot3\cdot4}{1\cdot3\cdot1}=8,$$

$$\begin{array}{|c|c|c|}\hline 3&4&5\\\hline\end{array}: \qquad \dim=\frac{3\cdot4\cdot5}{1\cdot2\cdot3}=10,$$

$$\begin{array}{|c|c|c|}\hline 3&4&5\\\hline 2&3&4\\\hline\end{array}: \qquad \dim=\frac{2\cdot3\cdot4\cdot3\cdot4\cdot5}{1\cdot2\cdot3\cdot2\cdot3\cdot4}=10.$$

Rules for tensor products: We now give rules for tensor products of irreducible representations represented by Young diagrams. As an example we take in $SU(3)$

$$\begin{array}{|c|}\hline \\\hline \\\hline\end{array}\otimes\begin{array}{|c|c|}\hline &\\\hline &\multicolumn{1}{c}{}\\\cline{1-1}\end{array}$$

i) Label the boxes of the second factor by row, e.g. a, b, c, ...:

$$\begin{array}{|c|c|}\hline &\\\hline &\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\longrightarrow\begin{array}{|c|c|}\hline a&a\\\hline b&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}.$$

ii) Add the boxes with the a's from the lettered diagram to the right-hand ends of the rows of the unlettered diagram to form all possible legitimate Young diagrams that have no more than one a per column.

$$\begin{array}{|c|c|c|}\hline &a&a\\\hline &\multicolumn{2}{c}{}\\\cline{1-1}\end{array},\qquad\begin{array}{|c|c|}\hline &a\\\hline &\multicolumn{1}{c}{}\\\cline{1-1} a&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}$$

Note that the diagram

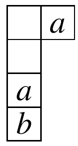
$$\begin{array}{|c|c|}\hline &a\\\hline &a\\\hline\end{array}$$

is not allowed since it has one column with two a's.

iii) Repeat the same with the b's, then with the c's, etc.

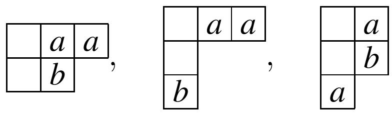
$$\begin{array}{|c|c|c|c|}\hline &a&a&b\\\hline &\multicolumn{3}{c}{}\\\cline{1-1}\end{array},\quad\begin{array}{|c|c|c|}\hline &a&a\\\hline &b&\multicolumn{1}{c}{}\\\cline{1-2}\end{array},\quad\begin{array}{|c|c|c|}\hline &a&a\\\hline &\multicolumn{2}{c}{}\\\cline{1-1} b&\multicolumn{2}{c}{}\\\cline{1-1}\end{array},\quad\begin{array}{|c|c|c|}\hline &a&b\\\hline &\multicolumn{2}{c}{}\\\cline{1-1} a&\multicolumn{2}{c}{}\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline &a\\\hline &b\\\hline a&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}$$

Note that the diagram

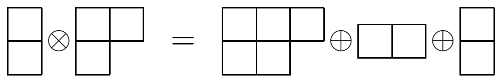


is not allowed for $SU(3)$, since it has more than 3 rows.

iv) A sequence of letter a, b, c, ... is admissible if at any point in the sequence at least as many a's have occured as b's, at least as many b's have occured as c's, etc. Thus *abcd* and *aabcb* are admissible sequences, while *abb* and *acb* are not. From the diagrams in step iii) throw away all diagrams in which the sequence of letters formed by reading right to left in the first row, then in the second row, etc., is not admissible. This leaves



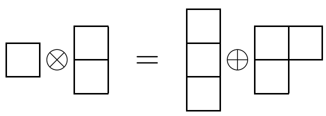
Removing complete columns of 3 boxes, we finally obtain



For the dimensions we have

$$\mathbf{3} \otimes \mathbf{8} \quad = \quad \mathbf{15} \oplus \mathbf{6} \oplus \mathbf{3}.$$

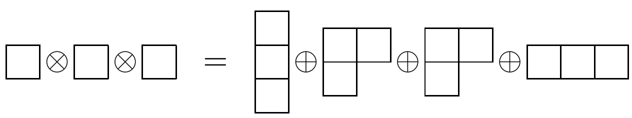
As a further example let us calculate in $SU(3)$ the tensor product of the fundamental representation with its complex conjugate representation:



For the dimensions we have

$$\mathbf{3} \otimes \bar{\mathbf{3}} \quad = \quad \mathbf{1} \oplus \mathbf{8}.$$

As a final example let us consider



For the dimensions we have

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \quad = \quad \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10}.$$

4 The classification of semi-simple Lie algebras

Recall: For a semi-simple Lie algebra $\mathfrak{g}$ of dimension n snd r we had the Cartan standard form

$$\begin{aligned} \left[H_i, H_j\right] &= 0, \\ \left[H_i, E_\alpha\right] &= \alpha_i E_\alpha, \end{aligned}$$

with generators H_i, $i = 1, ..., r$ as well as the generators E_α and $E_{-\alpha}$ with $\alpha = 1, ..., (n-r)/2$.

The generators H_i generate an Abelian sub-algebra of $\mathfrak{g}$. This sub-algebra is called the **Cartan sub-algebra** of $\mathfrak{g}$.

The r numbers α_i, $i = 1, ..., r$ are the components of the root vector $\vec{\alpha} = (\alpha_1, ..., \alpha_r)$.

We have already seen that if if $\vec{\alpha}$ and $\vec{\beta}$ are root vectors so is

$$\vec{\gamma} = \vec{\beta} - \frac{2\vec{\alpha} \cdot \vec{\beta}}{\alpha^2} \vec{\alpha}$$

Let us now put this a little bi more formally. For any root vector α we define a mapping W_α from the set of root vectors to the set of root vectors by

$$W_\alpha(\beta) = \vec{\beta} - \frac{2\vec{\alpha} \cdot \vec{\beta}}{\alpha^2} \vec{\alpha}$$

W_α can be described as the reflection by the plane Ω_α perpendicular to α. It is clear that this mapping is an involution: After two reflections one obtains the original root vector again. The set of all these mappings W_α generates a group, which is called the **Weyl group**.

Since W_α maps a root vector to another root vector, we have the following theorem:

Theorem: The set of root vectors is invariant under the Weyl group.

Actually, a more general result holds: We have seen that if $\vec{m}$ is a weight and if $\vec{\alpha}$ is a root vector then

$$W_\alpha(\vec{m}) = \vec{m} - \frac{2\vec{\alpha} \cdot \vec{m}}{\alpha^2} \vec{\alpha}$$

is again a weight vector. Therefore we can state that the following theorem:

Theorem: The set of weights of any representation of $\mathfrak{g}$ is invariant under the Weyl group.

The previous theorem is a special case of this one, as the root vectors are just the weights of the adjoint representation.

For the weights we defined an ordering. $\vec{m}$ is said to be higher than $\vec{m}'$ if the r^{th} component of $(\vec{m}-\vec{m}')$ is positive (if zero look at the $(r-1)^{\text{th}}$ component). This applies equally well to roots.

Definition: A root vectors $\vec{\alpha}$ is called positive, if $\vec{\alpha} > \vec{0}$.

Therefore the set of non-zero root vectors R decomposes into

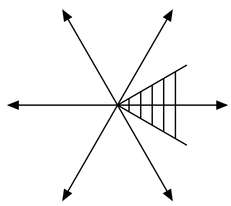
$$R = R^+ \cup R^-,$$

where R^+ denotes the positive roots and R^- denotes the negative roots.

Definition: The (closed) **Weyl chamber** relative to a given ordering is the set of points $\vec{x}$ in the r-dimensional space of root vectors, such that

$$2\frac{\vec{x}\cdot\vec{\alpha}}{\alpha^2} \geq 0 \qquad \forall \vec{\alpha} \in R^+.$$

Example: The Weyl chamber for $SU(3)$:



Let us further recall that if $\vec{\alpha}$ and $\vec{\beta}$ are root vectors then

$$\frac{2\vec{\alpha}\cdot\vec{\beta}}{|\alpha|^2} \quad \text{and} \quad \frac{2\vec{\alpha}\cdot\vec{\beta}}{|\beta|^2}$$

are integers. This restricts the angle between two root vectors to

$$0^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ, 120^\circ, 135^\circ, 150^\circ, 180^\circ$$

For $\theta = 30^\circ$ or $\theta = 150^\circ$ the ratio of the length of the two root vectors is

$$\frac{|\alpha|^2}{|\beta|^2} = 3 \text{ or } \frac{1}{3}.$$

For $\theta = 45^\circ$ or $\theta = 135^\circ$ the ratio of the length of the two root vectors is

$$\frac{|\alpha|^2}{|\beta|^2} = 2 \text{ or } \frac{1}{2}.$$

For $\theta = 60^\circ$ or $\theta = 120^\circ$ the ratio of the length of the two root vectors is

$$\frac{|\alpha|^2}{|\beta|^2} = 1.$$

Let us summarise: The root system R of a Lie algebra has the following properties:

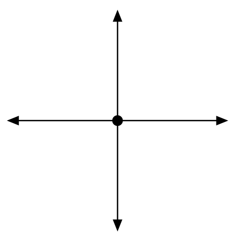
1. R is a finite set.
2. If $\vec{\alpha} \in R$, then also $-\vec{\alpha} \in R$.
3. For any $\vec{\alpha} \in R$ the reflection W_α maps R to itself.
4. If $\vec{\alpha}$ and $\vec{\beta}$ are root vectors then $2\vec{\alpha} \cdot \vec{\beta} / |\alpha|^2$ is an integer.

This puts strong constraints on the geometry of a root system. Let us now try to find all possible root systems of rank 1 and 2. For rank 1 the root vectors are one-dimensional and the only possibility is

A_1:

This is the root system of $SU(2)$. For rank 2 we first note that due to property (3) the angle between two roots must be the same for any pair of adjacent roots. It will turn out that any of the four angles 90°, 60°, 45° and 30° can occur. Once this angle is specified, the relative lengths of the roots are fixed except for the case of right angles. Let us start with the case $\theta = 90^\circ$. Up to rescaling the root system is

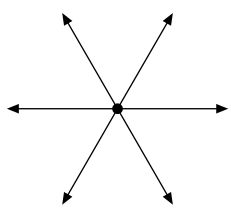
$A_1 \times A_1$:



This corresponds to $SU(2) \times SU(2)$. This group is semi-simple, but not simple. In general, the direct sum of two root systems is again a root system. A root system which is not a direct sum is called irreducible. An irreducible root system corresponds to a simple group. We would like to classify the irreducible root systems.

For the angle $\theta = 60^\circ$ we have

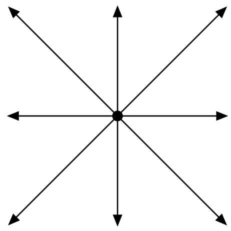
A_2



This is the root system of $SU(3)$.

For the angle $\theta = 45^\circ$ we have

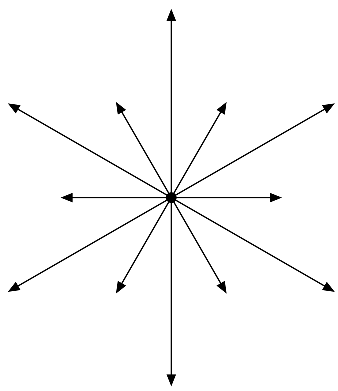
B_2:



This is the root system of $SO(5)$.

Finally, for $\theta = 30^\circ$ we have

G_2:



This is the root system of the exceptional Lie group G_2.

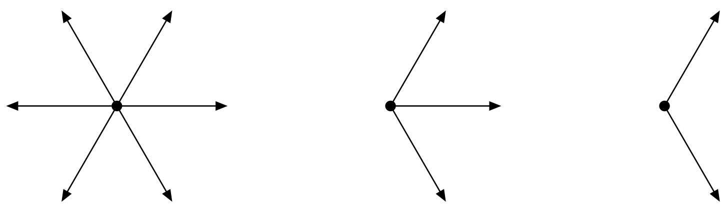
4.1 Dynkin diagrams

Let us try to reduce further the data of a root system. We already learned that with the help of an ordering we can divide the root vectors into a disjoint union of positive and negative roots:

$$R = R^+ \cup R^-.$$

Definition: A positive root vector is called **simple** if it is not the sum of two other positive roots.

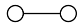
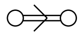
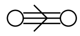
Example: For $SU(3)$ we have



root system positive roots simple roots

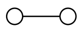
The angle between the two simple roots is $\theta = 120°$.

The Dynkin diagram of the root system is constructed by drawing one node $\circ$ for each simple root and joining two nodes by a number of lines depending on the angle θ between the two roots:

no lines		if $\theta = 90°$
one line		if $\theta = 120°$
two lines		if $\theta = 135°$
three lines		if $\theta = 150°$

When there is one line, the roots have the same length. If two roots are connected by two or three lines, an arrow is drawn pointing from the longer to the shorter root.

Example: The Dynkin diagram of $SU(3)$ is



4.2 The classification

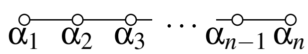
Semi-simple groups are a direct product of simple groups. For a compact group, all unitary representations are finite dimensional.
Real compact semi-simple Lie algebras $\mathfrak{g}$ are in one-to-one correspondence (up to isomorphisms) with complex semi-simple Lie algebras $\mathfrak{g}^{\mathbb{C}}$ obtained as the complexification of $\mathfrak{g}$. Therefore the classification of real compact semi-simple Lie algebras reduces to the classification of complex semi-simple Lie algebras.

Theorem: Two complex semi-simple Lie algebras are isomorphic if and only if they have the same Dynkin diagram.

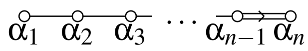
Theorem: A complex semi-simple Lie algebra is simple if and only if its Dynkin diagram is connected.

We have the following classification:

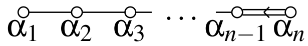
- $A_n \cong SL(n+1, \mathbb{C})$



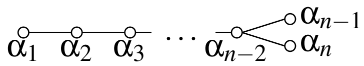
- $B_n \cong SO(2n+1, \mathbb{C})$



- $C_n \cong Sp(n,\mathbb{C})$

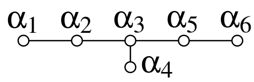


- $D_n \cong SO(2n,\mathbb{C})$

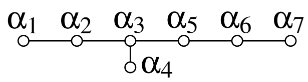


The exceptional groups are

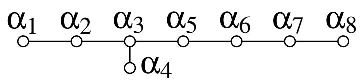
- E_6



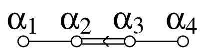
- E_7



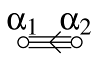
- E_8



- F_4



- G_2



Summary: The classical real compact simple Lie algebras are

$$\begin{aligned} A_n &= SU(n+1) \\ B_n &= SO(2n+1) \\ C_n &= Sp(n) \\ D_n &= SO(2n) \end{aligned}$$

The exceptional groups are

$$E_6, E_7, E_8, F_4, G_2$$

A semi-simple Lie algebra is determined up to isomorphism by specifying which simple summands occur and how many times each one occurs.

4.3 Proof of the classification

Recall: The root system R of a Lie algebra has the following properties:

1. R is a finite set.
2. If $\vec{\alpha} \in R$, then also $-\vec{\alpha} \in R$.
3. For any $\vec{\alpha} \in R$ the reflection W_α maps R to itself.
4. If $\vec{\alpha}$ and $\vec{\beta}$ are root vectors then $2\vec{\alpha} \cdot \vec{\beta} / |\alpha|^2$ is an integer.

With the help of an ordering we can divide the root vectors into a disjoint union of positive and negative roots:

$$R = R^+ \cup R^-.$$

A positive root vector is called simple if it is not the sum of two other positive roots.

The angle between two simple roots is either 90°, 120°, 135° or 150°

The Dynkin diagram of the root system is constructed by drawing one node ∘ for each simple root and joining two nodes by a number of lines depending on the angle θ between the two roots:

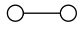
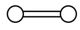
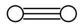
no lines	○ ○	if $\theta = 90°$
one line	○—○	if $\theta = 120°$
two lines	○⇒○	if $\theta = 135°$
three lines	○⇛○	if $\theta = 150°$

When there is one line, the roots have the same length. If two roots are connected by two or three lines, an arrow is drawn pointing from the longer to the shorter root.

Theorem: The only possible connected Dynkin diagrams are the ones listed in the previous section.

To prove this theorem it is sufficient to consider only the angles between the simple roots, the relative length do not enter the proof.

Such diagrams, without the arrows to indicate the relative lengths, are called **Coxeter diagrams**. Define a diagram of n nodes, with each pair connected by 0, 1, 2 or 3 lines, to be **admissible** if there are n independent unit vectors $\vec{e}_1$, ..., $\vec{e}_n$ in a Euclidean space with the angle between $\vec{e}_i$ and $\vec{e}_j$ as follows:

no lines		if $\theta = 90°$
one line		if $\theta = 120°$
two lines		if $\theta = 135°$
three lines		if $\theta = 150°$

Theorem: The only connected admissible Coxeter graphs are the ones of the previous section (without the arrows).

To prove this theorem, we will first prove the following lemmata:

(i) Any sub-diagram of an admissible diagram, obtained by removing some nodes and all lines to them, will also be admissible.

(ii) There are at most $(n-1)$ pairs of nodes that are connected by lines. The diagram has no loops.

(iii) No node has more than three lines to it.

(iv) In an admissible diagram, any string of nodes connected to each other by one line, with none but the ends of the string connected to any other nodes, can be collapsed to one node, and the resulting diagram remains admissible.

Proof of (i): Suppose we have an admissible diagram with n nodes. By definition there are n vectors $\vec{e}_j$, such that the angle between a pair of vectors is in the set

$$\{90°, 120°, 135°, 150°\}$$

Removing some of the vectors $\vec{e}_j$ does not change the angles between the remaining ones. Therefore any sub-diagram of an admissible diagram is again admissible.

Proof of (ii): We have

$$2\vec{e}_i \cdot \vec{e}_j \in \{0, -1, -\sqrt{2}, -\sqrt{3}\}$$

Therefore if $\vec{e}_i$ and $\vec{e}_j$ are connected we have $\theta > 90°$ and

$$2\vec{e}_i \cdot \vec{e}_j \quad \leq \quad -1.$$

Now

$$0 \quad < \quad \left(\sum_i \vec{e}_i\right) \cdot \left(\sum_j \vec{e}_i\right) \quad = \quad n + 2\sum_{i<j} \vec{e}_i \cdot \vec{e}_j \quad < \quad n - \#\text{ connected pairs}$$

Therefore

$$\#\,\text{connected pairs} \quad < \quad n.$$

Connecting n nodes with $(n-1)$ connections (of either 1, 2 or 3 lines) implies that there are no loops.

Proof of (iii): We first note that

$$\left(2\vec{e}_i \cdot \vec{e}_j\right)^2 \quad = \quad \#\,\text{number of lines between } \vec{e}_i \text{ and } \vec{e}_j.$$

Consider the node $\vec{e}_1$ and let $\vec{e}_i$, $i = 2, ..., j$ bet the nodes connected to $\vec{e}_1$. We want to show

$$\sum_{i=2}^{j} (2\vec{e}_1 \cdot \vec{e}_i)^2 \quad < \quad 4.$$

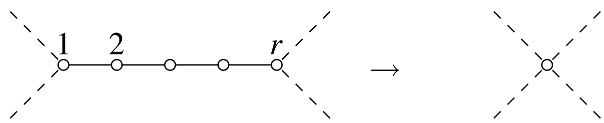
Since there are no loops, no pair of $\vec{e}_2$,...,$\vec{e}_j$ is connected. Therefore $\vec{e}_2$, ..., $\vec{e}_j$ are perpendicular unit vectors. Further, by assumption $\vec{e}_1$, $\vec{e}_2$,...,$\vec{e}_j$ are linearly independent vectors. Therefore $\vec{e}_1$ is not in the span of $\vec{e}_2$,...,$\vec{e}_j$. It follows

$$1 \quad = \quad (\vec{e}_1 \cdot \vec{e}_1)^2 \quad > \quad \sum_{i=2}^{j} (\vec{e}_1 \cdot \vec{e}_i)^2$$

and therefore

$$\sum_{i=2}^{j} (\vec{e}_1 \cdot \vec{e}_i)^2 < 1.$$

Proof of (iv):



If $\vec{e}_1$, ..., $\vec{e}_r$ are the unit vectors corresponding to the string of nodes as indicated above, then

$$\vec{e}' \quad = \quad \vec{e}_1 + ... + \vec{e}_r$$

is a unit vector since

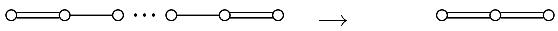
$$\begin{aligned} \vec{e}' \cdot \vec{e}' &= (\vec{e}_1 + ... + \vec{e}_r)^2 = r + 2\vec{e}_1 \cdot \vec{e}_2 + +2\vec{e}_2 \cdot \vec{e}_3 + ... + +2\vec{e}_{r-1} \cdot \vec{e}_r \\ &= r - (r-1) = 1. \end{aligned}$$

Further $\vec{e}'$ satisfies the same conditions with respect to the other vectors since $\vec{e}' \cdot \vec{e}_j$ is either $\vec{e}_1 \cdot \vec{e}_j$ or $\vec{e}_r \cdot \vec{e}_j$.

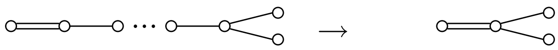
With the help of these lemmata we can now prove the original theorem:

From (iii) it follows that the only connected diagram with a triple line is G_2.

Further we cannot have a diagram with two double lines, otherwise we would have a sub-diagram, which we could contract as

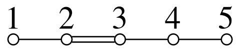


contradicting again (iii). By the same reasoning we cannot have a diagram with a double line and a triple node:



Again this contradicts (iii).

To finish the case with double lines, we rule out the diagram



Consider the vectors

$$\vec{v} = \vec{e}_1 + 2\vec{e}_2, \qquad \vec{w} = 3\vec{e}_3 + 2\vec{e}_4 + \vec{e}_5.$$

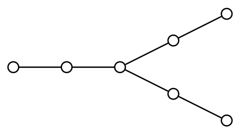
We find

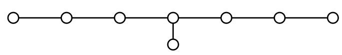
$$(\vec{v} \cdot \vec{w})^2 = 18, \quad |\vec{v}|^2 = 3, \quad |\vec{w}|^2 = 6.$$

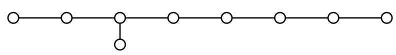
This violates the Cauchy-Schwarz inequality

$$(\vec{v} \cdot \vec{w})^2 \quad < \quad |\vec{v}|^2 \cdot |\vec{w}|^2 .$$

By a similar reasoning one rules out the following (sub-) graphs with single lines:







These sub-diagrams rules out all graphs not in the list of the previous section. To finish the proof of the theorem it remains to show that all graphs in the list are admissible. This is equivalent to show that for each Dynkin diagram in the list there exists a corresponding Lie algebra. (The simple root vectors of such a Lie algebra will then have automatically the corresponding angles of the Coxeter diagram.)

To prove the existence it is sufficient to give for each Dynkin diagram an example of a Lie algebra corresponding to it. For the four families A_n, B_n, C_n and D_n we have already seen that they correspond to the Lie algebras of $SU(n+1)$, $SO(2n+1)$, $Sp(n)$ and $SO(2n)$ (or $SL(n+1,\mathbb{C})$, $SO(2n+1,\mathbb{C})$, $Sp(n,\mathbb{C})$ and $SO(2n,\mathbb{C})$ in the complex case). In addition one can write down explicit matrix representations for the Lie algebras corresponding to the five exceptional groups E_6, E_7, E_8, F_4 and G_2.

Finally for the uniqueness let us recall the following theorem: Two complex semi-simple Lie algebras are isomorphic if and only if they have the same Dynkin diagram.