The Method of Epstein and Glaser I

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In order to review the first part of the Epstein Glaser talk given at the Hesselberg 2002 meeting the reader will be confronted with the original transparencies. The subject of the talk was a brief introduction to the renormalization method of Epstein and Glaser. It contains some historical notes, the inductive construction of the S-matrix, remarks on the splitting of causal distributions and some examples. The narration in form of the original talk is meant as a Gedankenstütze. The extent of readable summerisations is such that it needs not an additional one from me. In my eyes the original literature is some kind of classic and so I decided not to give a detailed review of the subject. In fact I will just give two references:

[1] H. Epstein and V. Glaser, Ann. Inst. H.Poincaré XIX (1973) 211

[2] G. Scharf, Finite Quantum Electrodynamics, Second Edition (1995), Springer Verlag

Theory of Renormalization and Regularization
24 February 2002 - 1 March 2002
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The Method of Epstein and Glaser I

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History

Stückelberg (1956)

Construction of the S-matrix by means of causality
(,,macroscopic" causality condition + unitarity)

Bogoliubov & Co (1959)

- simplification of causality condition
- adiabatic switching with test function

math. reason: S-matrix is operator-valued functional
not operator

asymptotic states are not simply generated by free fields

S as time-ordered exponential is ill defined!

Epstein, Glaser (1973) Regularization

- inductive construction of perturbation series by means of causality and translation invariance (unitarity is not used)

$\longrightarrow$ no divergences

$\longrightarrow$ Feynman rules hold on tree-level (not for closed loops)

decomposition of distributions with causal support into retarded and advanced parts. *distribution splitting*

- techniques of dispersion integrals (G. Källen)

- adiabatic switching with test function $g(x) \in \mathcal{S}(\mathbb{R}^4)$

- adiabatic limit $g \to 1$ at the end

Part I

Inductive construction of the S-matrix

Inductive construction of the S-matrix

multiply interaction by switching function $g(x)$, which vanishes rapidly for $x \to \pm\infty$

Aim: construct S-matrix as power series in g

$$
\begin{aligned}
S(g) &= \mathbf{1} + \sum_{n=1}^{\infty} \frac{1}{n!} \int d\,x_1 \dots d\,x_n T_n(x_1, \dots, x_n) g(x_1) \dots g(x_n) \\
&\doteq \mathbf{1} + T \\
S^{-1}(g) &= \mathbf{1} + \sum_{n=1}^{\infty} \frac{1}{n!} \int d\,x_1 \dots d\,x_n \tilde{T}_n(x_1, \dots, x_n) g(x_1) \dots g(x_n) \\
&\doteq (\mathbf{1} + T)^{-1} = \mathbf{1} + \sum_{r=1}^{\infty} (-T)^r
\end{aligned}
$$

$T_n(x_1, \dots, x_n) \in \mathcal{S}^*(\mathbb{M}^4)$ must be symmetric (otherwise the contribution vanishes)

The $\tilde{T}_n$ follow by expanding the r-th power of $-T$

$$
\tilde{T}_n(X) = \sum_{r=1}^{n} (-)^r \sum_{P_r} T_{n_1}(X_1) \dots T_{n_r}(X_r),
$$

where $X = \{x_1, \dots, x_n\}$ is a disordered set and P_r all partitions of X into r disjoint subsets

$$
X = X_1 \cup \dots \cup X_r, \quad X_j \neq 0, \quad X_i \cap X_j = \emptyset, \quad |X_j| = n_j
$$

In particular we have

$$
\tilde{T}_1(x) = -T_1(x)
$$

some properties of T_n:

$S(g)S(g)^{-1} = \mathbf{1}$, insert series expansion

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty} \sum_{n_1+n_2=n} \frac{1}{n_1! n_2!} \int d^4x_1 \dots d^4x_{n_1} d^4y_1 \dots d^4y_{n_2} \\ &\quad \times T(x_1, \dots, x_{n_1}) \tilde{T}(y_1, \dots, y_{n_2}) g(x_1) \dots g(x_{n_1}) g(y_1) \dots g(y_{n_2}) \\ &= \sum_{n=1}^{\infty} \sum_{P_2^0(X)} \frac{1}{n!} \int d^4x_1 \dots d^4x_n T(X_1) \tilde{T}(X_2) g(x_1) \dots g(x_n) \end{aligned}$$

where $P_2^0(X)$ is the partition of X into two sets (incl. $\emptyset$).

Therefore we get

$$\sum_{P_2^0(X)} T(X_1)\tilde{T}(X_2) = 0$$

analogously we get from $S(g)^{-1}S(g) = \mathbf{1}$

$$\sum_{P_2^0(X)} \tilde{T}(X_1)T(X_2) = 0$$

In scalar theories without degree of gauge freedom the S-matrix is unitary

$$S(g)^{-1} = S(g)^{\dagger} \quad \tilde{T}(X) = T(X)^{\dagger}$$

The distributions T_n inherit covariance from S, resp. the test-functions:

$$\begin{aligned} & (U(a,\Lambda)\varphi)(x) = \varphi(\Lambda^{-1}(x-a)) \qquad \varphi \in \mathcal{F} \\ \Rightarrow \qquad & U(a,\Lambda)S(g)U(a,\Lambda)^{-1} = S(g_{(a,\Lambda)}) \end{aligned}$$

where $g_{(a,\Lambda)}(x) = g(\Lambda^{-1}(x-a))$. So we get for T_n:

$$U(a,\Lambda)T_n(x_1,\dots,x_n)U(a,\Lambda)^{-1} = T_n(\Lambda x_1 + a, \dots, \Lambda x_n + a)$$

pivotal point: causality

suppose:

- support of $g(x)$ is decomposed $g(x) = g_1(x) + g_2(x)$
- $\exists$ reference frame with
  $x \in \text{supp}(g_1) \Rightarrow x^0 < 0$ and $x \in \text{supp}(g_2) \Rightarrow x^0 > 0$
  $\text{supp}g_2 > \text{supp}g_1$ (space-like separation)

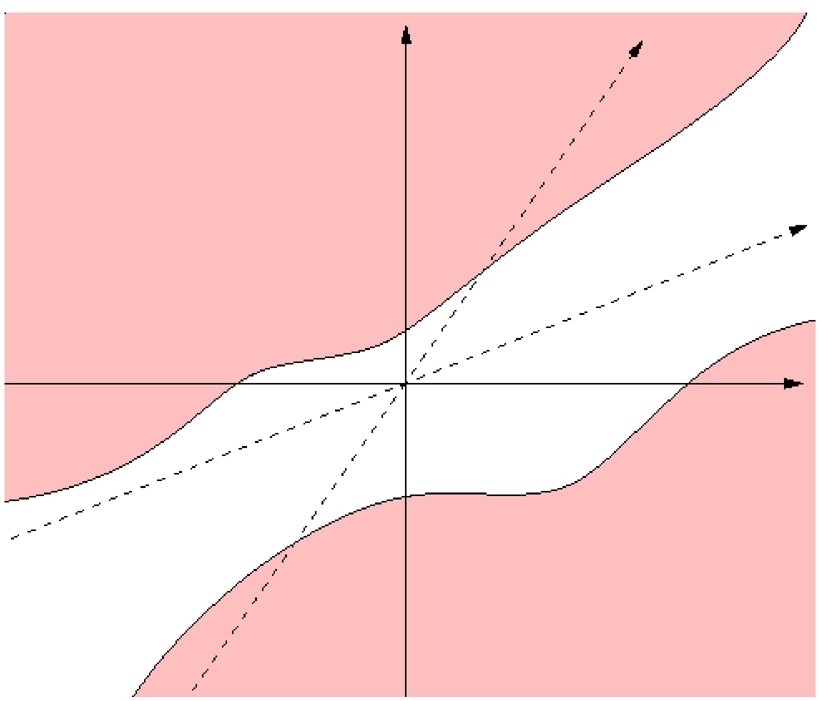
causality condition for the S-matrix:

$$S(g_1 + g_2) = S(g_2)S(g_1), \quad \text{if} \quad \text{supp}\, g_1 < \text{supp}\, g_2$$

,,what happens for $t < s$ is not influenced by what happens for $t > s$''

$$T_n(x_1, \ldots, x_n) = T_m(x_1, \ldots, x_m)T_{n-m}(x_{m+1}, \ldots, x_n),$$

$$\text{if} \quad \{x_1, \ldots, x_m\} > \{x_{m+1}, \ldots, x_n\}$$



temporal disjunctive supports

causal property

perturbative formulation of causality condition

$$S(g_1+g_2)=\sum_n \frac{1}{n!}\int d\,x_1\ldots d\,x_n T_n(x_1,\ldots,x_n)$$
$$\times(g_1(x_1)+g_2(x_1))\ldots(g_1(x_n)+g_2(x_n))$$

$\frac{n!}{m!(n-m)!}$ permutations of the switching functions

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{1}{m!(n-m)!}\int d\,x_1\ldots d\,x_n T_n(x_1,\ldots,x_n)$$
$$\times g_2(x_1)\ldots g_2(x_m)g_1(x_{m+1})\ldots g_1(x_n)$$

$$S(g_1)S(g_2)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{1}{m!(n-m)!}\int d\,x_1\ldots d\,x_n$$
$$\times T_m(x_1,\ldots,x_m)T_{n-m}(x_{m+1},\ldots,x_n)g_2(x_1)\ldots g_2(x_m)g_1(x_{m+1})\ldots g_1(x_n)$$

$$\Rightarrow T_n(x_1,\ldots,x_n)=T_m(x_1,\ldots,x_m)T_{n-m}(x_{m+1},\ldots,x_n)$$
$$\forall\{x_1,\ldots,x_m\}>\{x_{m+1},\ldots,x_n\}$$

inductively determination

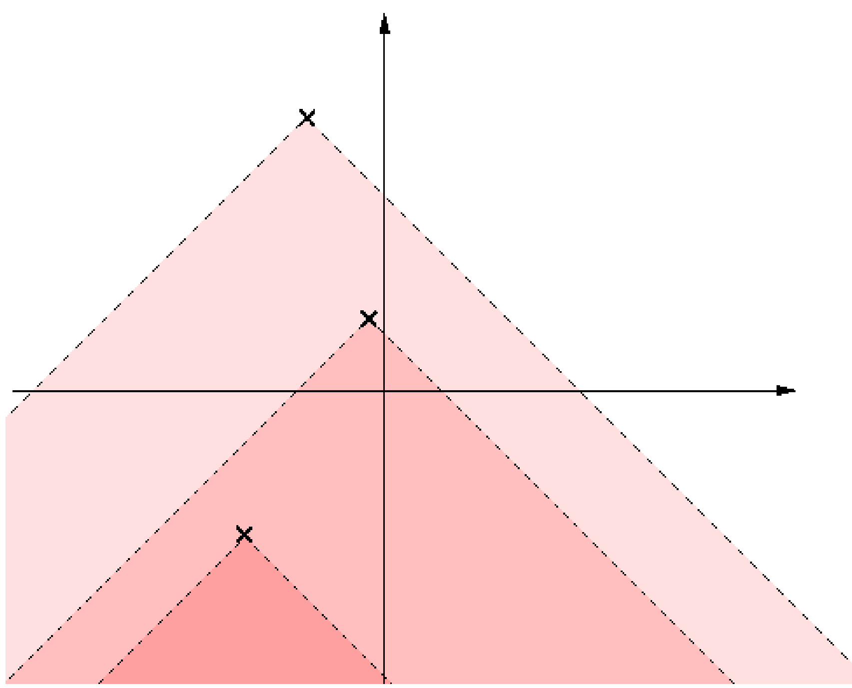
analogously it follows from $S(g_1+g_2)^{-1}=S(g_1)^{-1}S(g_2)^{-1}$

$$\tilde{T}_n(x_1,\ldots,x_n)=\tilde{T}_{n-m}(x_{m+1},\ldots,x_n)\tilde{T}_m(x_1,\ldots,x_m)$$

Notation:

$$X = \{x_j \in \mathbb{M}^4 | j = 1, \ldots, n\}$$

$$\{x_1, \ldots, x_{n'}\} > \{x_{n'+1}, \ldots, x_n\}$$



Three causal space-time points

all modules are developed to define the advanced and retarded distributions A_n and R_n.

Again: $\{x_1, \ldots, x_n\}$

$$A_{n+1}(X, x_{n+1}) \doteq \sum_{P_2^0(X)} \tilde{T}(X_1)T(X_2, x_{n+1})$$

$$R_{n+1}(X, x_{n+1}) \doteq \sum_{P_2^0(X)} T(X_1, x_{n+1})\tilde{T}(X_2)$$

decomposition of $X = X_1 \cup X_2$ containes $\emptyset$, therefore $T_{n+1}(X, x_{n+1})$ is part of the sum.

Define *primed* advanced and retarded distributions

$$\begin{aligned} A'_{n+1}(X, x_{n+1}) &\doteq \sum_{\substack{P_2^0(X) \\ X_1 \neq \emptyset}} \tilde{T}(X_1)T(X_2, x_{n+1}) \\ &\equiv A_{n+1}(X, x_{n+1}) - T_{n+1}(X, x_{n+1}) \\ R'_{n+1}(X, x_{n+1}) &\doteq \sum_{\substack{P_2^0(X) \\ X_2 \neq \emptyset}} T(X_1, x_{n+1})\tilde{T}(X_2) \\ &\equiv R_{n+1}(X, x_{n+1}) - T_{n+1}(X, x_{n+1}) \end{aligned}$$

$T_{n'}$ known? $\Rightarrow A'_{n+1}$, R'_{n+1} can be constructed.

A_{n+1}, R_{n+1} can't be constructed in this way, but D_{n+1}

$$D_{n+1} \doteq R_{n+1} - A_{n+1} \equiv R'_{n+1} - A'_{n+1}$$

 support properties filters R_{n+1} out from D_{n+1}

calculate then $T_{n+1} = R'_{n+1} - R_{n+1}$

Support properties of A, R and D

Theorem

$Y = P \cup Q$, $P \neq \emptyset$, $P \cap Q \neq \emptyset$, $|Y| = n_1 \leq n-1$ and $x \notin Y$

If $\{Q, x\} > P$, $|Q| = n_2$, then

$$R'_{n_1+1}(Y, x) = -T_{n_2+1}(Q, x) T_{n_1-n_2}(P)$$

If $\{Q, x\} < P$, then

$$A'_{n_1+1}(Y, x) = -T_{n_1-n_2}(P) T_{n_2+1}(Q, x)$$

Proof: use unitarity & causality

Corollary (support property 1)

$$\text{supp}\, R_{n_1+1}(Y, x) \subseteq \Gamma^{+}_{n_1+1}(x)$$

$$\text{supp}\, A_{n_1+1}(Y, x) \subseteq \Gamma^{-}_{n_1+1}(x)$$

where $\Gamma^{\pm}_{n}(x) = \{(x_1, \ldots, x_n) | x_j \in \overline{V^{\pm}}(x), \forall j = 1, \ldots, n\}$

and $\overline{V^{+}}(x) = \{y | (y-x)^2 \geq 0, y^0 \geq x^0\}$

resp. $\overline{V^{-}}(x) = \{y | (y-x)^2 \geq 0, y^0 \leq x^0\}$

,,$R_{n+1}(X, x)$ *vanishes if there is any point* $x' \in X$ *with* $x' < x$''

Theorem (support property 2)

If $n \geq 3$, then

$$\text{supp}\, D_n(x_1, \ldots, x_{n-1}, x_n) \subseteq \Gamma^{+}_{n-1}(x_n) \cup \Gamma^{-}_{n-1}(x_n)$$

For $n \leq 2$ this support property has to be shown explicitly.

Recipe - inductive construction

$T_m \quad 1 \leq m \leq n-1$ known

$\Downarrow$

construct advanced/retarded distributions

$$A'_n(x_1,\ldots,x_n) = \sum_{P_2} \tilde{T}_{n_1}(X) T_{n-n_1}(Y,x_n)$$

$$R'_n(x_1,\ldots,x_n) = \sum_{P_2} T_{n-n_1}(Y,x_n) \tilde{T}_{n_1}(X)$$

with $P_2 : \{x_1,\ldots,x_{n-1}\} = X \cup Y, \quad X \neq \emptyset, \quad n_1 = |X| \geq 1$

$\Downarrow$

add $X = \emptyset$:

$$A_n(x_1,\ldots,x_n) = A'_n(x_1,\ldots,x_n) + T_n(x_1,\ldots,x_n)$$
$$R_n(x_1,\ldots,x_n) = R'_n(x_1,\ldots,x_n) + T_n(x_1,\ldots,x_n)$$

$\Downarrow$

difference distribution

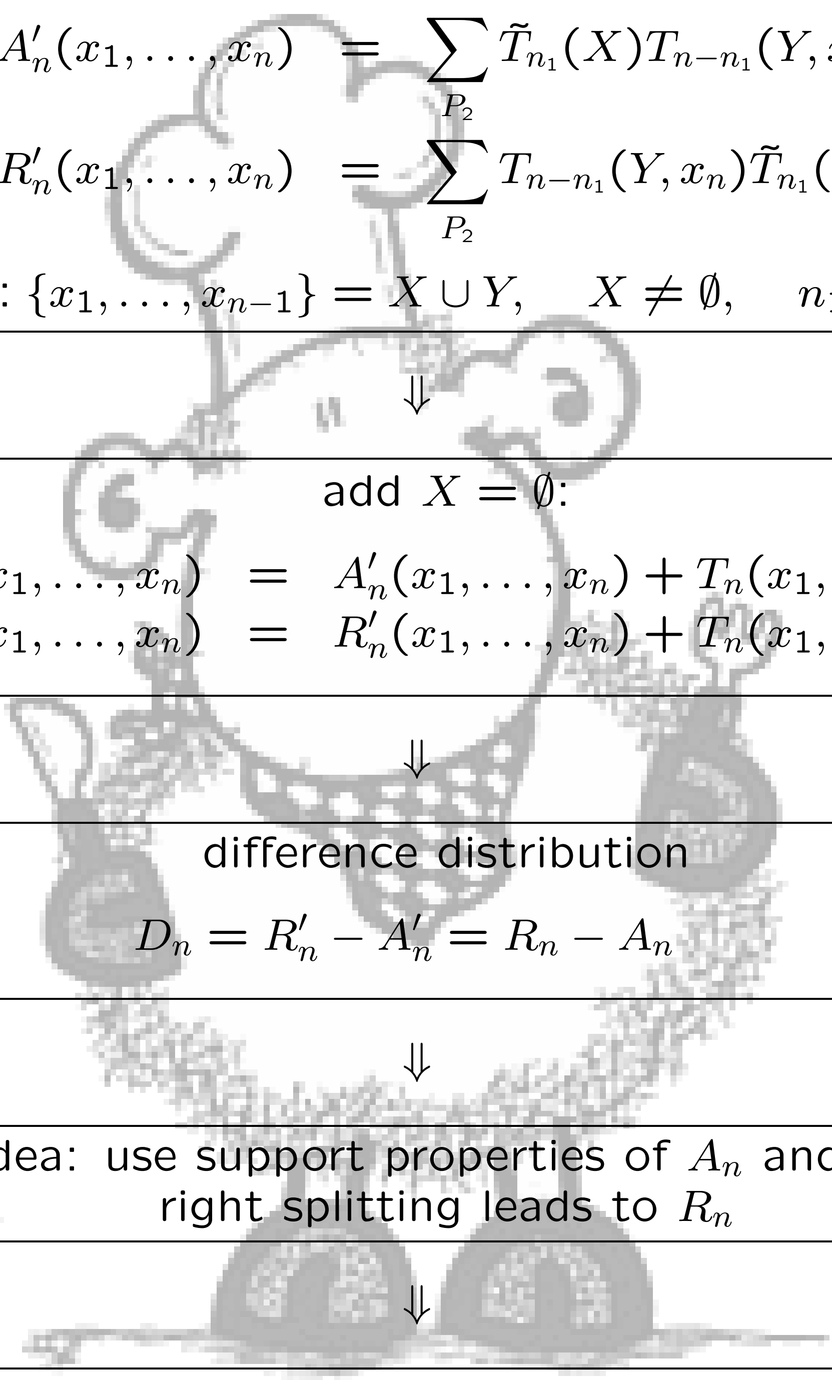
$$D_n = R'_n - A'_n = R_n - A_n$$

$\Downarrow$

idea: use support properties of A_n and R_n
right splitting leads to R_n

$\Downarrow$

T_n follows from $R_n = R'_n + T_n$
$\Rightarrow \quad T_n = R_n - R'_n$



Example

Provided first order $T_1(t)$ is given. Construct $T_2(t)$:

$$\begin{aligned} A'_2(t_1,t_2) &= \tilde{T}_1(t_1)T_1(t_2) = -T_1(t_1)T_1(t_2) \\ R'_2(t_1,t_2) &= T_1(t_2)\tilde{T}_1(t_1) = -T_1(t_2)T_1(t_1) \end{aligned}$$

$\hookrightarrow$ advanced and retarded functions

$$\begin{aligned} A_2(t_1,t_2) &= A'_2(t_1,t_2) + T_2(t_1,t_2) \\ R_2(t_1,t_2) &= R'_2(t_1,t_2) + T_2(t_2,t_1) \end{aligned} \quad (1)$$

$$D_2 = R_2 - A_2 = R'_2 - A'_2 \quad \text{is known}$$

R_2 and A_2 can be identified by support properties.

$$\begin{aligned} R_2(t_1,t_2) &= \Theta(t_1-t_2)D_2(t_1,t_2) \\ &= \Theta(t_1-t_2)\left(T_1(t_1)T_1(t_2) - T_1(t_2)T_1(t_1)\right) \end{aligned}$$

is up to $t_1 = t_2$ uniquely determined.

$T_2(t_1,t_2)$ follows for example from (1)

$$\begin{aligned} T_2(t_1,t_2) &= R_2(t_1,t_2) - R'_2(t_1,t_2) \\ &= \Theta(t_1-t_2)T_1(t_1)T_1(t_2) + \Theta(t_2-t_1)T_1(t_2)T_1(t_1) \\ &= \mathtt{T}\left\{T_1(t_1)T_1(t_2)\right\} \end{aligned}$$

time-ordered product, as it would be expected.

remember: S-matrix with usage of Schrödinger equation

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} dt_1 \ldots dt_n T\left[\tilde{V}(t_1)\ldots\tilde{V}(t_n)\right] \\ &= \mathtt{T}\exp\left[-\mathrm{i}\int_{-\infty}^{+\infty} \mathrm{d}t\tilde{\mathrm{V}}(\mathrm{t})\right] \end{aligned}$$

Quantum mechanical system described by $H = H^\dagger \in \mathcal{H}$

time evolution $\psi(t) = e^{-iHt}\psi_0$, with

$H = H_0 + V =$ free Hamiltonian $+$ interaction (short range)

The so-called wave operators exist as strong limits on $\mathcal{H}$

$$W_{\substack{\text{in}\\\text{out}}} = s - \lim_{t\to\mp\infty} e^{iHt}e^{-iH_0t}$$

In the case of a **time-dependent interaction** $V = V(t)$, we have

$$\psi(t) = U(t,s)\psi(s), \quad U(t,s)^\dagger = U(s,t), \quad U(t,s)U(s,r) = U(t,r)$$

Then the wave operators are defined as

$$W_{\substack{\text{in}\\\text{out}}} = s - \lim_{t\to\mp\infty} U(t,0)^\dagger e^{-iH_0t} = s - \lim_{t\to\mp\infty} U(0,t)e^{-iH_0t}$$

S-matrix, central object of scattering theory

$$S = W_{\text{out}}^\dagger W_{\text{in}} = \lim_{\substack{s\to-\infty\\t\to+\infty}} e^{iH_0t}U(t,s)e^{-iH_0s}$$

physical meaning:

- normalized initial asymptotic state φ at time $t = 0$
- first transformed to $s = -\infty$ by **free** dynamics
- evolve from $-\infty$ to $+\infty$ by full **interaction** dynamics
- transform back from $+\infty$ to $t = 0$ by **free**-dynamics

$S\varphi$ is therefore the outgoing scattering state, transformed to $t = 0$ by free time evolution

propability for transition from φ to ψ: $P(\varphi \to \psi) = |(\psi, S\varphi)|^2$

Consider $\psi(t)$ to be solution of the Schrödinger equation

$$i\dot{\psi}(t) = (H_0V(t))\psi(t) \qquad \hbar = 1$$

Go over to the interaction picture by substituting $\psi(t) = e^{-iH_0 t}\varphi(t)$. $\varphi(t)$ then satisfies

$$(*) \qquad i\dot{\varphi} = \tilde{V}(t)\varphi(t), \quad \text{where} \quad \tilde{V}(t) = e^{iH_0 t}V(t)e^{-iH_0 t}$$

In the interaction picture the S-matrix is just

$$S = \lim_{\substack{s\to-\infty \\ t\to+\infty}} \tilde{U}(t,s)$$

write $(*)$ as an integral equation

$$\varphi(t) = \varphi(s) - i\int_s^t dt_1 \tilde{V}(t_1)\varphi(t_1)$$

If $V(t)$ is a bounded operator, iteration leads to the Dysonseries

$$\varphi(t) = \underbrace{\left[1 + \sum_{n=1}^{\infty}(-i)^n \int_s^t dt_1 \int_s^{t_1} dt_2 \cdots \int_s^{t_{n-1}} dt_n \tilde{V}(t_1)\ldots\tilde{V}(t_n)\right]}_{\doteq U(t,s)} \varphi(s)$$

S-matrix is norm convergent and defines a unitary operator in $\mathcal{H}$

$$S = \sum_{n=0}^{\infty}(-i)^n \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_1 \tilde{V}(t_1)\ldots\tilde{V}(t_n)$$

domain of integraton: simplex in $\mathbb{R}^n$

extend to integral over cube

$$\begin{aligned} S &= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{+\infty} dt_1 \cdots \int_{-\infty}^{+\infty} dt_n \mathsf{T}\left[\tilde{V}(t_1)\ldots\tilde{V}(t_n)\right] \\ &\doteq \mathsf{T}\exp\left[-i\int_{-\infty}^{+\infty}\tilde{V}(t)\right] \end{aligned}$$

Part II

Splitting of Causal Distributions

space of distributions is per definition linear

QFT contains pointwise products of operator-valued distributions

continuation problem

Definition: A distribution $T \in \mathcal{S}'(\mathbf{R}^n)$ has *scale degree* δ at the point $x = 0$, if

$$\delta = \inf \{\delta' \in \mathbf{R} | \lambda^{\delta'} T(\lambda x) \stackrel{\lambda \to 0}{\longrightarrow} 0\}$$

The *singular order* is then given by $\omega = [\delta] - n$

Example 1. $\delta \in \mathcal{S}'(\mathbf{R^n})$: $\delta(\lambda x) = |\lambda|^{-n}\delta(x)$. Therefore $\delta = n$ and $\omega = 0$.

Example 2. $\frac{1}{(x^2)^3} \in \mathcal{S}'(\mathbf{R}^4)$ (third power of the Feynmanpropagator in scalar massless euclidean fieldtheory). Here: $\delta = 6$ and $\omega = 2$.

Proposition (Properties). $T \in \mathcal{S}'(\mathbf{R}^n)$, $\operatorname{scal\,deg}(T) = \delta$ and β is a multi-index. Then:

I $\operatorname{scal\,deg}(x^\beta T) = \delta - |\beta|$

II $\operatorname{scal\,deg}(D^\beta T) = \delta + |\beta|$

III $\operatorname{scal\,deg}(w) \leq 0$, $\operatorname{scal\,deg}(wT) \leq \delta$, $w \in \mathcal{S}(\mathbf{R}^n)$

IV $\operatorname{scal\,deg}(T_1 \otimes T_2) = \delta_1 + \delta_2$, since $\operatorname{scal\,deg}(T_i) = \delta_i$

Example 3. singular order of $\Box \frac{1}{x^2} \in \mathcal{S}'(\mathbf{R}^4)$ is 0.

proper definition of pointwise multiplication of distributions requires the improved version of the *micro-local scale degree*

Expand $D_n(X)$ in terms of free fields

$$D_n(X) = \sum_{l,l_i \leq k} d_n^{l_1 \ldots l_n} : \phi^{k-l_1}(x_1) \ldots \phi^{k-l_n}(x_n) :$$

Question: Splitting of $d_n^{l_1 \ldots l_n} = r_n - a_n$

$$\text{with} \quad \begin{array}{l} \operatorname{supp} r_n \subseteq \Gamma^+_{n-1}(x_n) \\ \operatorname{supp} a_n \subseteq \Gamma^-_{n-1}(x_n)\,? \end{array}$$

$\omega \leq -1$, that is $\delta \leq n-1$: no continuation problem

$\omega \geq 0$, that is $\delta \geq n$: continuation can be constructed, if testfunctions vanish up to the degree ω at 0

continuation in general case via projection into subspace.

Definition (W-Operation).

$\mathcal{S}^\omega(\mathbf{R}^n)$: space of testfunction which vanish up to the degree ω at 0.

Operator, which projects from $\mathcal{S}(\mathbf{R}^n)$ into $\mathcal{S}^\omega(\mathbf{R}^n)$:

$$W_{(\omega;w)} : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}^\omega(\mathbf{R}^n), \quad \varphi \to W_{(\omega;w)}\varphi$$

$$\left(W_{(\omega;w)}\varphi\right)(x) = \varphi(x) - w(x) \sum_{|\alpha| \leq \omega} \frac{x^\alpha}{\alpha!} \left(D^\alpha \frac{\varphi}{w}\right)(0)$$

,,Taylor substraction''

The right splitting procedure

$\omega \leq -1$

$$\langle r_m(x), \varphi \rangle \doteq \langle d, \prod_{j=1}^{m-1} \theta(x_j^0 - x_m^0) \varphi(x) \rangle$$

$\omega \geq 0$

$$\langle r_m(x), \varphi \rangle \doteq \langle d, \prod_{j=1}^{m-1} \theta(x_j^0 - x_m^0) W \varphi(x) \rangle$$

$\uparrow$

W-operation

$\Rightarrow$ guaranties $r_m \subseteq \Gamma^+_{m-1}(x_n)$

ambiguity: $r_m - \tilde{r}_m = \sum_{|a| \leq \omega} \tilde{c}_a D^a \delta(x)$

$\tilde{c}_a$'s have to be fixed by physical requirements.

Example - Regularization

$$\int_{-\infty}^{+\infty} \frac{1}{x}\varphi(x)dx \qquad \text{illdefined at} \quad \pm\infty, 0$$

$$f(x) = \frac{1}{x} \qquad \text{def.} \quad \forall x \in \mathbb{R}\setminus\{0\}$$

look for distribution $\tilde{f}$:

$$(\tilde{f},\varphi) = \int_{-\infty}^{+\infty} \tilde{f}(x)\varphi(x)dx, \quad \varphi \in \mathcal{S}(\mathbb{R})$$

Gelfand-Shilov regularization

$$\int_{-\infty}^{+\infty} dx\frac{1}{x}\varphi(x) = \int_0^{+\infty} dx\frac{\varphi(x)-\varphi(-x)}{x}$$

Epstein-Glaser:

$$\begin{aligned}
\int_{-\infty}^{+\infty} dx\frac{1}{x}\varphi(x) &= \int_{-\infty}^{+\infty} dx\frac{1}{x}\left(\varphi(x) - w(x)\varphi(0)\right) \quad \text{Taylor-substraction} \\
&= \int_0^{\infty} dx\frac{1}{x}\left(\varphi(x) - w(x)\varphi(0)\right) - \int_0^{\infty} dx\frac{1}{x}\left(\varphi(-x) - w(-x)\varphi(0)\right) \\
&= \underbrace{\int_0^{\infty} dx\frac{1}{x}\left(\varphi(x) - \varphi(-x)\right)}_{\text{GS-Reg.}} - \varphi(0)\underbrace{\int_0^{\infty} dx\frac{1}{x}\left(w(x) - w(-x)\right)}_{\text{GS-Reg. with } w(x)}
\end{aligned}$$

That is

$$\begin{aligned}
\left(\frac{1}{x},\varphi\right)_{\text{EG}} &= \left(\frac{1}{x},\varphi\right)_{\text{GS}} - \varphi(0)\left(\frac{1}{x},w\right)_{\text{GS}} \qquad \forall\varphi \\
&= \left(\frac{1}{x},\varphi\right)_{\text{GS}} - (\delta,\varphi)\underbrace{\left(\frac{1}{x},w\right)_{\text{GS}}}_{c(w) \quad \text{constant}}
\end{aligned}$$

For example:

Selfenergy of the electron:

el. potential

$$\Delta\phi = -4\pi\rho = 4\pi e\delta \quad \Rightarrow \quad \phi = -\frac{e}{r} \in \mathcal{S}'(\mathbf{R}^3)$$

el. field

$$\mathbf{E} = -\nabla\phi = -\frac{e}{r^2} \in \mathcal{S}'(\mathbf{R}^3)$$

Since supp$(\mathbf{E}) = \{0\}$:

$$\mathbf{E}^2 = \frac{e^2}{r^4} \in \mathcal{S}'(\mathbf{R}^3 \setminus \{0\}) \tag{2}$$

singular order $\omega = 1$. $\Rightarrow$ Continuation to all test functions with W-operation and energy density $U = \mathbf{E}^2$ can be defined

$$\langle U, \varphi \rangle \doteq \langle \mathbf{E}^2, W_{(1;w)}\varphi \rangle \tag{3}$$

electron in rest: self-energy $E = 1/(4\pi)\langle U, 1\rangle$. Infrared-divergence $\Rightarrow$ choose $\varphi \equiv 1$ and $w \equiv 1$. Then

$$E = \frac{1}{4\pi}\langle U, W_{(1;1)}1\rangle + C^0 = C^0 \tag{4}$$

C^0 can be quoted, if one claims that the electron mass is only of electronic magnetic nature, that is $E = mc^2$.