Modern Physics Letters A, Vol. 17, No. 40 (2002) 2611–2615

THE PRESENTATION OF THE QUANTUM ALGEBRA OF OBSERVABLES OF THE CLOSED BOSONIC STRING IN $1 + 3$ DIMENSIONS: THE EXACT QUANTIZED GENERATING RELATIONS OF ORDERS $\hbar^6$ AND $\hbar^7$

G. HANDRICH, C. PAUFLER*, J. B. TAUSK and M. WALTER

Physikalisches Institut, Albert-Ludwigs-Universität,
Hermann-Herder-Str. 3, D-79104 Freiburg i. Br., Germany
**cornelius.paufler@physik.uni-freiburg.de*

Received 15 December 2002

We proceed with the investigation of a method of quantization of the observable sector of closed bosonic strings. For the presentation of the quantum algebra of observables the construction cycle concerning elements of order $\hbar^6$ has been carried out. We have computed the quantum corrections to the only generating relation of order $\hbar^6$. This relation is of spin-parity $J^P = 0^+$. We found that the quantum corrections to this relation break the semidirect splitting of the classical algebra into an Abelian, infinitely generated subalgebra $\mathfrak{a}$ and a non-Abelian, finitely generated subalgebra $\mathfrak{U}$. We have established that there are no ("truly independent") generating relations of order $\hbar^7$.

Keywords: String theory; observable sector; presentation.

PACS Nos.: 11.25.Hf, 11.25.Sq

1. Introduction

Closed bosonic strings can be described by an infinite set of parametrization invariant generators and their algebra. On the classical level they form a Poisson algebra.[1] Without loss of generality we restrict ourselves to the investigation of the subalgebra of "right-movers" $\mathfrak{h}^-$, or rather to its quantum counterpart $\hat{\mathfrak{h}}^-$. Furthermore, we consider the case of *massive* strings moving in $(1 + 3)$-dimensional Minkowski space.

In the following it is assumed that the reader is familiar with the content of Ref. 1 (for a recent review, cf. Ref. 2). Here, we will adhere to the notation and terminology employed there.

$\hat{\mathfrak{h}}^-$ carries an $\mathbb{N}$-gradation in the $\frac{\hbar}{2\pi\alpha'}$-scaling dimension of its elements (α' denoting the inverse string tension), for short in "powers of $\hbar$". Moreover, $\hat{\mathfrak{h}}^-$ carries

*Corresponding author

a representation of the $O(3)$ symmetry of the target space. Hence, its elements can be organized in powers of $\hbar$ and in so(3) spin-parity multiplets, denoted by J^P. The spin of a given multiplet in the algebra will be indicated by a subscript. In particular, commutator ($[\cdot\,,\cdot]$) and anticommutator ($\{\cdot\,,\cdot\}$) operations can be organized to yield so(3) multiplets again: let $[\cdot\,,\cdot]_j$ and $\{\cdot\,,\cdot\}_j$ be the respective spin j multiplets.

In Ref. 1, an algorithm has been given for the construction of a presentation both of the classical and of the quantum algebras, starting from the spin operator $\hat{\mathcal{J}}_1$ of order $\hbar$ and the elements $\hat{\mathcal{S}}_1$, $\hat{\mathcal{S}}_2$, $\hat{\mathcal{T}}_2$ and $\hat{\mathcal{B}}_0^{(1)}$ of power $\hbar^2$, and subsequently increasing cycle by cycle the $\hbar$-power by one, introducing new generators $\hat{\mathcal{B}}_0^{(2l+1)}$of the subalgebra $\hat{\mathfrak{a}}$ in every even order $\hbar^{2l+2}$, $l = (0), 1, 2, \ldots$. Recall from Ref. 1, p. 25 that $\hat{\mathcal{J}}_1$, $\hat{\mathcal{S}}_1$, $\hat{\mathcal{S}}_2$, $\hat{\mathcal{T}}_2$ and $\hat{\mathcal{B}}_0^{(2l+1)}$ are of spin-parity 1^+, 1^-, 2^-, 2^+ and 0^+, respectively.

The consistency up to all orders in $\hbar$ of the results of the previous cycles of this algorithm, in particular their agreement with the guiding principle of structural correspondence between the classical and the quantum algebras, has been proved recently by Meusburger and Rehren.[2]

On the classical level, the set of elements $\mathcal{J}_1$, $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{T}_2$ generates a Poisson subalgebra $\mathfrak{U}$ of $\mathfrak{h}^-$, while the infinite set of elements $\mathcal{B}_0^{(2l+1)}$, $l = 0, 1, \ldots$, forms an Abelian subalgebra $\mathfrak{a}$ of $\mathfrak{h}^-$. Moreover, it has been observed previously that the two Poisson subalgebras $\mathfrak{a}$ and $\mathfrak{U}$ of $\mathfrak{h}^-$, at least up to a "grade" (Ref. 1, p. 27) which would correspond to order $\hbar^8$ in the respective quantum algebras, are disjoint and that the Poisson bracket action of the first two elements of $\mathfrak{a}$, $\hat{\mathcal{B}}_0^{(1)}$ and $\hat{\mathcal{B}}_0^{(3)}$, on the elements of $\mathfrak{U}$ is semidirect. Among other things our analysis below will clarify the relevance of this observation for the quantum algebras, with $\hat{\mathfrak{a}}$ and $\hat{\mathfrak{U}}$ denoting the respective quantum counterparts.

2. Presentation of the Results

We have computed the cycle of "degree 5" (Ref. 1, pp. 39 and 40) concerning all the relations in $\hat{\mathfrak{U}}$ of order $\hbar^6$.

For this, the only new generating relation of order $\hbar^6$ had to be taken into account. The classical version of this relation has been calculated by one of us (G.H.), after its existence had been established in the course of the computation of the *quantum* action of the generator $\hat{\mathcal{B}}_0^{(3)}$ on the elements of $\hat{\mathfrak{U}}$.[3] Its quantum corrections are constrained by the requirement that the commutator with the generators produce no new relations of lower order in $\hbar$ without a classical counterpart.

The complete quantum relation is given in the appendix of an extended version of the article at hand.[a] The fact that this relation appears to be a rather

[a] math-ph/0210024.

long expression depends crucially on the basis chosen[b] in order $\hbar^3$ and higher. Its existence and content, however, are independent of the choice of such a basis.

For the definition of the quantum relation, the most general correction term has been added, introducing 14 parameters of order $\hbar$, 12 of order $\hbar^2$ and 2 of order $\hbar^4$ [sic!]. Note that there are no elements of the algebra $\hat{\mathfrak{h}}^-$ that are of order $\hbar^3$ and spin-parity 0^+ (Ref. 1, p. 27); hence, there cannot be any $\hbar^3$ correction terms. Among the added terms which are linearly independent of each other, there are exactly two, $\hat{\mathcal{B}}_0^{(3)}$ and $\hat{\mathcal{B}}_0^{(1)}$, from the infinitely generated Abelian subalgebra $\hat{\mathfrak{a}}$. Their pertinent coefficients will be denoted by x_1 and x_2. They are of respective order $\hbar^2$ and $\hbar^4$. At first sight, two more $\hbar^4$ correction terms (other than $\hat{\mathcal{B}}_0^{(3)}$) seem to be allowed, to wit the anticommutators $\{\{\hat{\mathcal{J}}_1, \hat{\mathcal{J}}_1\}_0, \hat{\mathcal{B}}_0^{(1)}\}_0$ and $\{\hat{\mathcal{B}}_0^{(1)}, \hat{\mathcal{B}}_0^{(1)}\}_0$. However, a computation reveals that e.g. their commutators with $\hat{\mathcal{S}}_1$ yield two polynomially independent elements not contained in $\hat{\mathfrak{U}}$ which cannot be compensated by other commutators, due to the semidirect product structure valid at lower orders of $\hbar$. On the other hand, by the already established correspondence between classical and quantum algebras in the subleading orders of $\hbar$, there is no room for additional relations (below $\hbar^6$). Therefore, the respective coefficients of the above-mentioned terms must vanish in the quantum relation.

Following the construction algorithm with the help of computer-algebraic software[c] we have obtained the following results:

(1) The cycle of induction into the $J^P = 0^+$, $\hbar^7$-sector shows that vanishing coefficients have to be assigned to all of the 14 correction terms of order $\hbar$, in accordance with the $\mathbb{Z}_2$-grading of the quantum corrections that has been observed in Ref. 1, p. 48.
(2) All of the relations of order $\hbar^7$ can be obtained by the induction of relations of order $\hbar^4$ and lower. This observation is based on the fact that the number of independent elements of a given $\hbar$-power and spin-parity is known explicitly

[b]In our algorithm, this basis is determined by the order in which the elements of a given power in $\hbar$ are to be arranged. For instance, for pure commutator basis elements, apart from the general techniques developed by M. Hall[4] for the presentation of free Lie algebras, we have, roughly speaking, chosen the following scheme. (The treatment of pure anticommutators and mixtures of commutators and anticommutators proceeds similarly — it can be deduced from the arrangement of the terms in the $\hbar^6$, 0^+ relation.) First, sort the pure commutator terms by the number of times each of the generators $\hat{\mathcal{T}}_2$, $\hat{\mathcal{S}}_2$, $\hat{\mathcal{S}}_1$ occurs separately in each term. Introduce the ordering $\hat{\mathcal{T}}_2 > \hat{\mathcal{S}}_2 > \hat{\mathcal{S}}_1$. Arrange the expressions with equal numbers of occurring $\hat{\mathcal{T}}_2, \hat{\mathcal{S}}_2, \hat{\mathcal{S}}_1$ according to the length of the longest uninterrupted iterated commutator, i.e. of the longest sequence of the form (we suppress the spin subscripts) $[\cdots[[g, g'], g''], \ldots, g^{(n)}]$, where the $g^{(i)}$ denote generators. Then, arrange all pure commutators according to the following rules (let f and g be pure commutators): $[f, g]_l > [f', g']_{l'}$ if $f > f'$ or if $f = f'$ and $g > g'$, or if $f = f'$, $g = g'$ and $l > l'$.

[c]We are using Mathematica routines that have been developed by C. Nowak and G. Handrich in the course of their lower $\hbar$-order investigation.[3] In order to cope with the sheer numbers of equations and variables occurring at the present $\hbar$-order — in some sectors, systems of as many as 10^4 equations in around 10^4 variables arise — we have translated part of the routines into the symbolic manipulation language FORM.[5]

and the fact that there do not exist generating relations of order $\hbar^5$. In other words, granting the knowledge of the $\hat{\mathcal{B}}_0^{(1)}$- and $\hat{\mathcal{B}}_0^{(3)}$-actions on $\hat{\mathcal{S}}_1$, $\hat{\mathcal{S}}_2$, $\hat{\mathcal{T}}_2$ (the former being given in Ref. 1, the latter in Ref. 3), the set of relations given in Ref. 1 plus the sole generating relation of order $\hbar^6$ gives a complete description of the finitely generated $\hat{\mathfrak{U}}$-part of the algebra of observables at least up to order $\hbar^7$.

(3) Regarding the requirement of correspondence, no restrictions on the (as yet undetermined) parameters f, g_1, g_2 have been found. (These parameters have been introduced in the course of definition of quantum relations of lower power of $\hbar$ in Ref. 1, p. 44, and Ref. 3. Should their values ever be determined in cycles of higher power of $\hbar$, they will assume rational values, cf. Ref. 1, p. 47.) However, all of the 12 surviving $\hat{\mathfrak{U}}$-contributions to the quantum correction of the $\hbar^6$, 0^+ relation can be expressed in terms of f and g_2. More precisely, the 11 coefficients of order $\hbar^2$ are linear functions of f, and the remaining $\hbar^4$ coefficient is quadratic in f and linear in g_2.

(4) The parameter x_1 that describes the contribution of $\hat{\mathcal{B}}_0^{(3)}$ in the quantum relation turns out to be *nonzero*. The numerical value for x_1 can be inferred from the quantum relation given in the extended version of this paper. The fact that x_1 is nonzero means that the semidirect decomposition of the classical algebra $\mathfrak{h}^-$ into a sum of the Abelian subalgebra $\mathfrak{a}$ and the finitely generated subalgebra $\mathfrak{U}$ *cannot* be carried over to the quantum case.

At hindsight, one could have anticipated that elements from the Abelian subalgebra $\hat{\mathfrak{a}}$ had to be included in the quantum corrections to the new classical generating relation: observe that the generating relation of order $\hbar^6$ is the first encounter with such a relation of characteristic $J^P = 0^+$ and that, in addition, it is of *even* order in powers of $\hbar$, invoking, in accordance with the $\mathbb{Z}_2$-grading, 0^+ quantum corrections of *even* orders in powers of $\hbar$. The generators of the Abelian subalgebra, $\hat{\mathcal{B}}_0^{(2l+1)}$, $l = 0, 1, 2, \ldots$ are all of this type. Thus they must be considered as candidates for the quantum corrections. Moreover, for every generator[d] $\hat{\mathcal{S}}_1$, $\hat{\mathcal{S}}_2$, $\hat{\mathcal{T}}_2$ there is exactly one way to induce such a relation together with its most general quantum correction into $J^P = 1^-, 2^-, 2^+$ relations of $\hbar$-order increased by one, respectively. Further, for the present case, notice that there are only three Lie-type basis elements (i.e. basis elements that can be expressed as pure commutators of the generators) contained in the vector space of characteristic $\hbar^4$, $J^P = 0^+$, not belonging to the subalgebra $\hat{\mathfrak{a}}$. Hence, upon the above inductions — apart from the generator $\hat{\mathcal{B}}_0^{(3)}$ of the Abelian algebra $\hat{\mathfrak{a}}$ — only these three elements give rise to linear combinations of Lie-type basis elements of order $\hbar^5$ (besides elements not of Lie-type). In other words, at most *three* linearly independent such combinations contained in the pertinent vector spaces can be obtained in turn by this procedure. However,

[d]In this argument, we disregard the generator $\hat{\mathcal{B}}_0^{(1)}$ because it commutes with the $\hat{\mathcal{B}}_0^{(2l+1)}$-quantum corrections by construction.

this number is rather poor in comparison with the numbers 16, 15 and 14 of Lie-type elements of the respective vector spaces. Small wonder that assistance by the generator $\hat{\mathcal{B}}_0^{(3)}$ of the Abelian subalgebra is badly needed as an additional quantum correction, which, upon the inductions by $\hat{\mathcal{S}}_1$, $\hat{\mathcal{S}}_2$ and $\hat{\mathcal{T}}_2$, each time gives rise to one additional independent linear combination of Lie-type basis elements of order $\hbar^5$, among other things.

Because of the ambiguity in the definition of the subalgebra $\hat{\mathfrak{a}}$ (Ref. 6, see also Ref. 2), one might hope that by a different choice for the generators $\hat{\mathcal{B}}_0^{(2l+1)}$ of $\hat{\mathfrak{a}}$ the (quantum) breaking of the semidirect splitting (of the classical algebra $\mathfrak{h}^-$) could be avoided. This is in fact impossible, since already for the present case every different choice for the generator $\hat{\mathcal{B}}_0^{(3)}$ in the $\hbar^4$-power quantum corrections to the generating $\hbar^6$-0^+-relation results in a mere redefinition of some of the coefficients of the $\hat{\mathfrak{U}}$-terms in this part of the relation.

(5) The other parameter, x_2, characterizing the $\hat{\mathcal{B}}_0^{(1)}$ dependence, is given as a function linear both in f and in g_1.

Using a *concrete* realization of the algebra of observables, Meusburger and Rehren found the variables f, g_1 and g_2 to take on certain rational numbers, the latter property in agreement with the results of Ref. 1. Insertion of these values into the formula for x_2 reveals a *nonvanishing* contribution of $\hat{\mathcal{B}}_0^{(1)}$ to the quantum corrections as well.

A conservative *a priori* estimate would suggest that the parameters of the quantum corrections would be roughly tenfold overdetermined. As yet, the reason for the fact that they are *not* overdetermined has not been identified in terms of special properties of the generating relations and the actions of $\hat{\mathcal{B}}_0^{(2l+1)}$, $l = 0, 1, 2, \ldots$. Instead, Meusburger and Rehren[2] give a reason in terms of the properties of an embedding algebra.

Acknowledgments

This project has been carried out under the supervision of K. Pohlmeyer. The authors wish to thank him for thorough discussions on the interpretation of the results and for many suggestions concerning the manuscript. This work was supported in part by the DFG-Forschergruppe Quantenfeldtheorie, Computeralgebra und Monte-Carlo Simulation.

References

1. K. Pohlmeyer, *Ann. Phys.* (*Leipzig*) **8**, 1, 19–50 (1999).
2. C. Meusburger and K.-H. Rehren, “Algebraic quantization of the closed bosonic string”, math-ph/0202041.
3. G. Handrich and C. Nowak, *Ann. Phys.* (*Leipzig*) **8**, 1, 51–54 (1999); extended version: hep-th/9807231.
4. M. Hall, Jr., *Proc. Amer. Math. Soc.* **1**, 575 (1950).
5. J. A. M. Vermaseren, “New features of FORM”, math-ph/0010025.
6. K. Pohlmeyer and K.-H. Rehren, *Commun. Math. Phys.* **114**, 55 (1988).