A VERTICAL EXTERIOR DERIVATIVE IN MULTISYMPLECTIC GEOMETRY AND A GRADED POISSON BRACKET FOR NONTRIVIAL GEOMETRIES

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(Received February 9, 2000 — Revised May 18, 2000)

A vertical exterior derivative is constructed which is needed for a graded Poisson structure on multisymplectic manifolds over nontrivial vector bundles. In addition, the properties of the Poisson bracket are proved and first examples are discussed.

1. Introduction

In [7] a geometrical framework to handle field theories over manifolds in a finite-dimensional geometry is proposed. This mathematical setting appears in the literature under the name multisymplectic geometry, De Donder–Weyl theory, Hamilton–Cartan formalism, and covariant field theory ([6, 3, 5], further [8, 9]). The basic idea is to treat the space coordinates of a given field theory as additional evolution parameters. Thus, there is a finite number of variables (the field and its first derivatives) that evolve in space-time rather than a curve in an infinite-dimensional vector space of field configurations. As shown in [18, 16], one can incorporate the field equations and the Noether theorem [15] in this formulation, but in order to find a corresponding quantum field theory—at least in the sense of a formal deformation [1, 4]—one has to formulate the dynamics of the classical theory in terms of Poisson brackets first.

Kanatchikov [11, 12] has constructed such a bracket for trivial vector bundles over orientable manifolds. In the nontrivial case he used "vertical exterior derivative" which plays a central rôle in the construction is not globally defined (the resulting bracket, however, does not depend on the coordinate system used). What is needed is a derivative in vertical directions that in particular has square zero. A first guess would be to use a connection and take an expression like $dv^A \wedge \nabla_A$ with ∇ being a covariant derivative and dv^A being vertical. The condition that its square gives zero is then equivalent to the flatness of ∇ along fibres. As the fibres under consideration are vector spaces, one would indeed expect that it is possible to construct such a covariant derivative. This construction constitutes the main part of this paper.

The remaining part of this article is organised as follows. In the first section a short overview over the multisymplectic approach is given. Then, with the help of a covariant derivative that is flat along the fibres of phase space, the already mentioned vertical exterior derivative is constructed and discussed. Next the Poisson structure is given and the defining properties are proved. Finally, mechanics as the case of a trivial (vector) bundle over a one-dimensional base manifold (i.e., the time axis $\mathbb{R}$) is recovered and the scalar field case is considered.

Appendix contains some well-known facts about connections viewed as sections of jet bundles and the construction of the already mentioned covariant derivative on the multisymplectic phase space.

2. From variational principles to multisymplectic geometry

In field theory, solutions of the field equations are stationary points of the action functional

$$L[\varphi] = \int_{\mathcal{M}} \mathcal{L}\left(\varphi(x), \nabla\varphi(x)\right)\, d^{n+1}x,$$

where $\mathcal{M}$ is some $(n+1)$-dimensional parameter space (e.g. space-time), $\nabla\varphi$ is the gradient of the field φ and $\mathcal{L}$ is the Lagrange density.

In general, φ is a section of a vector bundle $\pi : \mathcal{V} \to \mathcal{M}$. $T\varphi : T\mathcal{M} \to T\mathcal{V}$ fulfills $T\pi \circ T\varphi = T\,\mathrm{id}_{\mathcal{M}}$ and thus[1] defines an element of $\mathfrak{J}^1\mathcal{V}$, the first jet bundle of $\mathcal{V}$ [17, 14]. Using a linear connection

$$\Gamma : \mathcal{V} \to \mathfrak{J}^1\mathcal{V}$$

we obtain an isomorphism

$$i_\Gamma : \left(\mathfrak{J}^1\mathcal{V}\right)_v \to \left(\mathcal{V} \otimes T^*\mathcal{M}\right)_{\pi(v)}$$

for all v in $\mathcal{V}$, where in addition we have used $(\mathfrak{V}\mathcal{V})_v \cong \mathcal{V}_{\pi(v)}$ for vector bundles $\mathcal{V}$ and their vertical tangent bundles $\mathfrak{V}\mathcal{V}$. In particular, we find

$$i_\Gamma \circ T_x\varphi \circ \xi(x) = \nabla_\xi \varphi(x), \tag{1}$$

for ∇ denoting the covariant derivative corresponding to Γ and ξ being a tangent vector on $\mathcal{M}$. This will be needed in Section 4.

Now the Lagrange density can be interpreted as a mapping

$$\mathcal{L} : \mathfrak{J}^1\mathcal{V} \to \Lambda^{n+1}T^*\mathcal{M}, \qquad L[\varphi] = \int_{\mathcal{M}} \mathcal{L} \circ j^1\varphi,$$

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[1] Usually, the first jet bundle of a vector bundle $(\mathcal{M}, \pi, \mathcal{V})$ is defined to be the set of all equivalence classes at a point of $\mathcal{M}$ of local sections, where equivalence means equal function value and first derivatives. But this can be viewed as a tangent map from $T\mathcal{M}$ to $T\mathcal{V}$ having the stated property. Further, such a tangent map defines how to lift $T\mathcal{M}$ (horizontally) at every point of $\mathcal{V}$, which is equivalent to having a connection. Hence, a connection defines a map $\mathcal{V} \to \mathcal{J}^1\mathcal{V}$, which turns the affine bundle $\mathcal{J}^1\mathcal{V}$ into a vector bundle over $\mathcal{V}$.

where $j^1\varphi(x) = T_x\phi \in \left(\mathfrak{J}^1\mathcal{V}\right)_{\varphi(x)}$ is the first jet prolongation of $\varphi \in \Gamma(\mathcal{V})$. Stationary points of L correspond to solutions of the Euler–Lagrange equations, which in local coordinates[2] (x^i, v^A, v_i^A) of $\mathfrak{J}^1\mathcal{V}$ read (cf. [16])

$$\frac{\partial \mathcal{L}}{\partial v^A} \circ j^1\varphi - \frac{\partial}{\partial x^i}\left(\frac{\partial \mathcal{L}}{\partial v_i^A} \circ j^1\varphi\right) = 0. \tag{2}$$

Now we want to formulate the theory of what we shall call a phase space. Since $\mathfrak{J}^1\mathcal{V}$ is not a vector bundle but an affine bundle, one chooses the dual $\left(\mathfrak{J}^1\mathcal{V}\right)^*$ to be the bundle of affine mappings from $\mathfrak{J}^1\mathcal{V}$ to $\Lambda^{n+1}T^*\mathcal{M}$. Thus, coordinates (x^i, v^A, v_i^A) on $\mathfrak{J}^1\mathcal{V}$ induce coordinates (x^i, v^A, p, p_A^i) on $\left(\mathfrak{J}^1\mathcal{V}\right)^*$. One can show (see [7], Chapter 2B) that $(\mathfrak{J}^1\mathcal{V})^*$, being a vector bundle over $\mathcal{V}$ (it inherits a vector space structure from the target space $\Lambda^{n+1}T^*\mathcal{M}$), is canonically isomorphic to $\mathcal{Z} \subset \Lambda^{n+1}T^*\mathcal{V}$, where

$$\mathcal{Z}_v = \{z \in \Lambda^{n+1}T^*\mathcal{V}_v \mid i_V i_W z = 0 \ \forall V,\ W \in (\mathfrak{V}\mathcal{V})_v\}, \qquad \mathcal{Z} = \bigcup_{v\in\mathcal{V}} \mathcal{Z}_v.$$

Furthermore, on $\Lambda^{n+1}T^*\mathcal{V}$ there is a canonical $(n+1)$-form Θ_Λ defined by

$$\Theta_\Lambda(z)(u_1, \ldots, u_{n+1}) = z(T\pi_{\mathcal{V}\Lambda}u_1, \ldots, T\pi_{\mathcal{V}\Lambda}u_{n+1}),$$

where $z \in \Lambda^{n+1}T^*\mathcal{V}$, $u_1, \ldots, u_{n+1} \in T_z\Lambda^{n+1}T^*\mathcal{V}$, $\pi_{\mathcal{V}\Lambda} : \Lambda^{n+1}T^*\mathcal{V} \to \mathcal{V}$. Using the embedding $i_{\Lambda\mathcal{Z}} : \mathcal{Z} \to \Lambda^{n+1}T^*\mathcal{V}$, we obtain an $(n+1)$-form on $\mathcal{Z}$,

$$\Theta = i^*_{\Lambda\mathcal{Z}}\Theta_\Lambda, \tag{3}$$

which will be called canonical $(n+1)$-form thereafter. There is a canonical $(n+2)$-form Ω on $\mathcal{Z}$, too,

$$\Omega = -d\Theta.$$

Using coordinates (x^i, v^A, p, p_A^i), one finds

$$\Theta = p_A^i\, dv^A \wedge \left(\partial_{x^i} \lrcorner\, d^{n+1}x\right) + p\, d^{n+1}x, \qquad \Omega = dv^A \wedge dp_A^i \wedge \left(\partial_{x^i} \lrcorner\, d^{n+1}x\right) - dp \wedge d^{n+1}x,$$

where $d^{n+1}x = dx^1 \wedge \cdots \wedge dx^{n+1}$. Now we are in the position to reformulate (2). As the first step we define a covariant Legendre transform for $\mathcal{L}$:

$$\begin{aligned} &\mathbb{F}\mathcal{L} : \mathfrak{J}^1\mathcal{V} \ni \gamma \mapsto \mathbb{F}\mathcal{L}(\gamma) \in \left(\mathfrak{J}^1\mathcal{V}\right)^* \cong \mathcal{Z}, \\ &\mathbb{F}\mathcal{L}(\gamma) : \mathfrak{J}^1\mathcal{V} \ni \gamma' \mapsto \mathcal{L}(\gamma) + \frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{L}\left(\gamma + \epsilon(\gamma' - \gamma)\right) \in \Lambda^{n+1}T^*\mathcal{M}. \end{aligned} \tag{4}$$

In the coordinates as above it takes the form

[2] When working in local coordinates of $\mathfrak{J}^1\mathcal{V}$ we will use the following convention. Small Latin indices sum over the base manifold directions, that is i, j, k run from 1 to $n+1$, if not specified otherwise. Capital Latin characters as A, B, C, D run from 1 to N which is the dimension of a fiber of $\mathcal{V}$.

$$\mathcal{L} = L(x^i, v^A, v_i^A)\, d^{n+1}x, \qquad p_A^i = \frac{\partial L}{\partial v_i^A}, \qquad p = L - \frac{\partial L}{\partial v_i^A} v_i^A. \tag{5}$$

Using $\mathbb{F}\mathcal{L}$ we can pull back the canonical $(n+1)$-form Ω to obtain the so-called Cartan form $\Theta_{\mathcal{L}}$,

$$\Theta_{\mathcal{L}} = (\mathbb{F}\mathcal{L})^* \Theta.$$

One can show ([7], Theorem 3.1) that the Euler–Lagrange equations (2) are equivalent to

$$\left(j^1\varphi\right)^* (i_W \Omega_{\mathcal{L}}) = 0 \qquad \forall W \in T\mathfrak{J}^1\mathcal{V},$$

where

$$\Omega_{\mathcal{L}} = -d\Theta_{\mathcal{L}} = (\mathbb{F}\mathcal{L})^* \Omega.$$

3. A vertical exterior derivative

To simplify the notation let us denote the multisymplectic phase space $(\mathfrak{J}^1\mathcal{V})^*$ by $\mathcal{P}$. In what follows we will need a mapping that is in some sense the vertical part of the exterior derivative on $\mathcal{P}$. In particular, it must have square zero. Whereas the derivation along fibres of $\mathcal{P} \to \mathcal{M}$ can be defined without additional data, the space of vertical forms as a subspace of arbitrary forms cannot be so defined[3]. This is due to the fact that one needs to specify what is *not* vertical if one is looking for the dual of vertical vectors. For this, one needs a connection in the bundle $\mathcal{P}$ over $\mathcal{M}$. This is considered in Appendix A. With the help of this connection we can split $T_p\mathcal{P}$ into horizontal and vertical components for each point p of $\mathcal{P}$. In the local coordinates[4] (x^i, v^A, p_A^i, p) we have a basis $(\mathfrak{e}^{*\alpha}_{(p)}, \mathfrak{e})$, $\alpha = i,\ A,\ {}^{\,i}_{A}$, of $T_p^*\mathcal{P}$ that is dual to a basis $(\mathfrak{e}_\alpha(p), \mathfrak{e})$ of $T_p\mathcal{P}$. The detailed definition of the latter is explained in Appendix. In the coordinates as above,

$$\begin{aligned} &\mathfrak{e}^{*i}_{(p)} = dx^i, \qquad \mathfrak{e}^{*A}_{(p)} = dv^A + \Gamma^A_{iB}(\pi(p))\, v^B\, dx^i, \\ &\mathfrak{e}^{*\,{}^{i}_{A}}_{(p)} = dp^{\,{}^{i}_{A}} + (\Lambda^i_{kj}\delta^B_A - \Gamma^B_{kA}\delta^i_j)\, p^{\,{}^{k}_{B}}\, dx^j, \qquad \mathfrak{e}^*_{(p)} = dp. \end{aligned} \tag{6}$$

Using the duality between $T\mathcal{P}$ and $T^*\mathcal{P}$, we obtain a covariant derivative D^* on $T\mathcal{P}$, in particular

$$\begin{aligned} (D^*{}_{\mathfrak{e}_M}\mathfrak{e}^{*N})(\mathfrak{e}_\rho)(p) &= -\mathfrak{e}^{*N}(D_{\mathfrak{e}_M}\mathfrak{e}_\rho)(p) = 0, \\ (D^*{}_{\mathfrak{e}_M}\mathfrak{e}^{*i})(\mathfrak{e}_\rho)(p) &= -\mathfrak{e}^{*i}(D_{\mathfrak{e}_M}\mathfrak{e}_\rho)(p) = 0 \end{aligned}$$

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[3] One can, however, define the space of vertical forms canonically, but in what follows we need the wedge product of a vertical form and an arbitrary one. For this, one needs an embedding of vertical forms in the space of forms, which in turn requires the use of a connection.

[4] When working in coordinates of $\mathcal{P}$, we will use the following convention which is similar to the one for coordinates on $\mathfrak{J}^1\mathcal{V}$. Small Latin indices sum over the base manifold directions, that is, i, j, k run from 1 to $n+1$ if not specified otherwise. Capital Latin characters as A, B, C, D run from 1 to N which is the dimension of a fibre of $\mathcal{V}$. Small Greek indices can be both base manifold and $\mathcal{V}$-fibre and dual jet bundle indices, i.e. $\rho, \sigma, \tau = i, A,\ {}^{\,i}_{A}$. Finally, capital letters from M onwards stand for both $A, B \ldots$ and ${}^{\,i}_{A}, {}^{\,j}_{B}, \ldots$.

for all fibre indices $M, N = A, {}^{i}_{A}$ and all indices ρ. Thus for every $\alpha(p) = \frac{1}{l!}\alpha_{\rho_1\cdots\rho_l(p)}\mathfrak{e}^{*\rho_1}_{(p)} \wedge \cdots \wedge \mathfrak{e}^{*\rho_l}_{(p)} \in \Omega^l\mathcal{P} = \Gamma(\Lambda^l T^*\mathcal{P})$ the mapping[5]

$$d^V = \left(\mathfrak{e}^{*M}_{(p)} \wedge D^*{}_{\mathfrak{e}_M}\right) : \Omega^l\mathcal{P} \to \Omega^{l+1}\mathcal{P}$$

fulfills ($M, N = A, {}^{i}_{A}$ for $i = 1, \ldots, n$, $A = 1, \ldots, N$, $\rho_l = i, A, {}^{i}_{A}$)

$$\begin{aligned}
\left(d^V\right)^2 \alpha(p) &= \left(d^V\right)^2 \frac{1}{l!}\alpha_{\rho_1\cdots\rho_l(p)}\mathfrak{e}^{*\rho_1}_{(p)} \wedge \cdots \wedge \mathfrak{e}^{*\rho_l}_{(p)} \\
&= \left(\mathfrak{e}^{*M}_{(p)} \wedge D^*{}_{\mathfrak{e}_M}\right)\left(\mathfrak{e}^{*N}_{(p)} \wedge D^*{}_{\mathfrak{e}_N}\right)\frac{1}{l!}\alpha_{\rho_1\cdots\rho_l(p)}\mathfrak{e}^{*\rho_1}_{(p)} \wedge \cdots \wedge \mathfrak{e}^{*\rho_l}_{(p)} \\
&= \frac{1}{l!}\left(\mathfrak{e}^{*M}_{(p)} \wedge D^*{}_{\mathfrak{e}_M}\right)\left(\mathfrak{e}_N\,\alpha_{\rho_1\cdots\rho_l}\right)_{(p)}\mathfrak{e}^{*N}_{(p)} \wedge \mathfrak{e}^{*\rho_1}_{(p)} \wedge \cdots \wedge \mathfrak{e}^{*\rho_l}_{(p)} \\
&\quad + \frac{1}{l!}\left(\mathfrak{e}^{*M}_{(p)} \wedge D^*{}_{\mathfrak{e}_M}\right)\sum_{k=1}^{l}\alpha_{\rho_1\cdots\rho_l(p)}\mathfrak{e}^{*N}_{(p)} \wedge \mathfrak{e}^{*\rho_1}_{(p)} \wedge \cdots \wedge \underbrace{D^*{}_{\mathfrak{e}_N}\mathfrak{e}^{*\rho_k}}_{=0} \wedge \cdots \wedge \mathfrak{e}^{*\rho_l}_{(p)} \\
&= \frac{1}{l!}\left(\mathfrak{e}_M\,\mathfrak{e}_N\,\alpha_{\rho_1\cdots\rho_l}\right)_{(p)}\mathfrak{e}^{*M}_{(p)} \wedge \mathfrak{e}^{*N}_{(p)} \wedge \mathfrak{e}^{*\rho_1}_{(p)} \wedge \cdots \wedge \mathfrak{e}^{*\rho_l}_{(p)} \\
&\quad + \frac{1}{l!}\left(\mathfrak{e}_N\,\alpha_{\rho_1\cdots\rho_l}\right)_{(p)}\mathfrak{e}^{*M}_{(p)} \wedge \underbrace{D^*{}_{\mathfrak{e}_M}\mathfrak{e}^{*N}}_{=0} \wedge \mathfrak{e}^{*\rho_1}_{(p)} \wedge \cdots \wedge \mathfrak{e}^{*\rho_l}_{(p)} \\
&\quad + \frac{1}{l!}\sum_{k=1}^{l}\left(\mathfrak{e}_N\,\alpha_{\rho_1\cdots\rho_l}\right)_{(p)}\mathfrak{e}^{*M}_{(p)} \wedge \mathfrak{e}^{*N}_{(p)} \wedge \mathfrak{e}^{*\rho_1}_{(p)} \wedge \ldots \wedge \underbrace{D^*{}_{e_N}\mathfrak{e}^{*\rho_k}}_{=0} \wedge \ldots \wedge \mathfrak{e}^{*\rho_l}_{(p)} \\
&= \frac{1}{2l!}\left([\mathfrak{e}_M, \mathfrak{e}_N]\,\alpha_{\rho_1\cdots\rho_l}\right)_{(p)}\mathfrak{e}^{*M}_{(p)} \wedge \mathfrak{e}^{*N}_{(p)} \wedge \mathfrak{e}^{*\rho_1}_{(p)} \wedge \cdots \wedge \mathfrak{e}^{*\rho_l}_{(p)} \\
&= 0,
\end{aligned}$$

that is, $\left(d^V\right)^2 = 0$. This justifies the name vertical exterior derivative.

3.1. Poincaré lemma for d^V

LEMMA 3.1 (Poincaré lemma for d^V). *Let $\alpha \in \Omega^r\mathcal{P}$ with $d^V\alpha = 0$. Then for every $p \in \mathcal{P}$ there exists a neighbourhood U_p and an $(r-1)$-form β such that $\alpha\restriction\mathcal{U}_p = d^V\beta$.*

Proof: As fibres of $\mathcal{P} \to \mathcal{M}$ are contractible and d^V, restricted to such a fibre, acts like the exterior derivative, this is a consequence of the Poincaré lemma. In detail, let $m = \pi(p)$ and $\mathcal{U}$ be a neighbourhood of m such that $\mathcal{P}\restriction\mathcal{U}$ is trivial. Now

[5] This mapping is a globally defined version of the vertical differential used by Kanatchikov in [11, 12].

let $\mathcal{U}_p = \pi^{-1}(\mathcal{U})$. On $\mathcal{U}_p$ we can choose a basis $(\mathfrak{e}^{*\alpha}_{(p)}, \mathfrak{e}^{*i}_{(p)})$ of $T^*\mathcal{P}|\mathcal{U}_p$ as above (in what follows we will omit the point p when writing a covector). Then we have

$$\alpha(p) = \sum_{l=0}^{r} \alpha_l(p),$$

where α_l is of the form

$$\alpha_l(p) = \frac{1}{r!}\alpha_{M_1\cdots M_l i_{l+1}\cdots i_r}(p)\, \mathfrak{e}^{*M_1} \wedge \cdots \wedge \mathfrak{e}^{*M_l} \wedge \mathfrak{e}^{*i_{l+1}} \wedge \cdots \wedge \mathfrak{e}^{*i_r}.$$

As $\mathfrak{e}^{*M_1} \wedge \cdots \wedge \mathfrak{e}^{*M_l} \wedge \mathfrak{e}^{*i_{l+1}} \wedge \cdots \wedge \mathfrak{e}^{*i_r}$ and $\mathfrak{e}^{*M_1} \wedge \cdots \wedge \mathfrak{e}^{*M_j} \wedge \mathfrak{e}^{*i_{j+1}} \wedge \cdots \wedge \mathfrak{e}^{*i_r}$ are linearly independent for $j \neq l$, $d^V\alpha = 0$ implies

$$d^V \alpha_l = 0, \qquad \forall l = 1, \ldots, r.$$

Furthermore, we see that

$$d^V \alpha_l(p) = 0 \quad \Leftrightarrow \quad d^V \alpha_{l,i_{l+1}\cdots i_r}(p) = 0, \qquad \forall i_{l+1}, \cdots, i_r = 1, \ldots, n,$$

where

$$\alpha_l(p) = \frac{1}{(r-l)!}\alpha_{l,i_{l+1}\cdots i_r}(p) \wedge \mathfrak{e}^{*i_{l+1}} \wedge \cdots \wedge \mathfrak{e}^{*i_r}.$$

Now, if we restrict the $\alpha_{l,i_{l+1}\cdots i_r}$ to a fixed fibre $\mathcal{P}_m$ of $\mathcal{P} \to \mathcal{M}$, applying d^V corresponds to the exterior derivative on that space. As the fibre under consideration is a vector space, it follows that

$$\alpha_{l,i_{l+1}\cdots i_r}\restriction_{\mathcal{Z}_m} = d^V \beta^m_{(l-1),i_{l+1}\cdots i_r},$$

and hence

$$\begin{aligned}
\alpha(p) = \sum_{l=0}^{r} \alpha_l(p) &= \sum_{l=0}^{r} \frac{1}{(r-l)!}\alpha_{l,i_{l+1}\cdots i_r}(p) \wedge \mathfrak{e}^{*i_{l+1}} \wedge \cdots \wedge \mathfrak{e}^{*i_r} \\
&= \sum_{l=0}^{r} \frac{1}{(r-l)!}\left(d^V \beta^{\pi(p)}_{(l-1),i_{l+1}\cdots i_r}\right) \wedge \mathfrak{e}^{*i_{l+1}} \wedge \cdots \wedge \mathfrak{e}^{*i_r} \\
&= \sum_{l=0}^{r} \frac{1}{(r-l)!} d^V \left(\beta^{\pi(p)}_{(l-1),i_{l+1}\cdots i_r} \wedge \mathfrak{e}^{*i_{l+1}} \wedge \cdots \wedge \mathfrak{e}^{*i_r}\right) \\
&= d^V \beta(p),
\end{aligned}$$

where

$$\beta(p) = \sum_{l=0}^{r} \frac{1}{(r-l)!}\beta^{\pi(p)}_{(l-1),i_{l+1}\cdots i_r} \wedge \mathfrak{e}^{*i_{l+1}} \wedge \cdots \wedge \mathfrak{e}^{*i_r}.$$

□

4. Field equations

As already mentioned, the multisymplectic phase space $\mathcal{P}$ of a given field theory is chosen to be the affine dual of the first jet bundle $\mathcal{J}^1\mathcal{V}$, but the field equations (2) are formulated on $\mathcal{J}^1\mathcal{V}$ itself. Hence, similarly to ordinary mechanics, one uses the covariant Legendre transformation (4) to reformulate the theory. For this, let us assume that the middle equation of (5) can be rearranged so that the variables v_i^A can be expressed in terms of $(x^i, v^A, \overset{i}{p}_A)$. In other words, we require

$$\det\left(\frac{\partial^2 L}{\partial v_i^A \partial v_j^B}\right) \neq 0, \quad v_i^A = \varphi_i^A(x^i, v^A, \overset{i}{p}_A).$$

Then the Lagrange density L, (5), becomes a function over phase space,

$$\tilde{L}(x^i, v^A, \overset{i}{p}_A) = L(x^i, v^A, \varphi_i^A(x^i, v^A, \overset{i}{p}_A))$$

and we obtain the covariant Hamiltonian

$$H(x^i, v^A, \overset{i}{p}_A) = \tilde{L}(x^i, v^A, \overset{i}{p}_A) - \overset{i}{p}_A \varphi_i^A(x^i, v^A, \overset{i}{p}_A). \tag{7}$$

Using this, the generalized Hamiltonian equations

$$\frac{\partial H}{\partial v^A} = \frac{\partial \overset{i}{p}_A}{\partial x^i}, \quad \frac{\partial H}{\partial \overset{i}{p}_A} = -\frac{\partial v^A}{\partial x^i}, \tag{8}$$

are equivalent to the Euler–Lagrange equations (2), ([16], Chapter 4.2). Note, however, that H is not a function but (7) describes rather a subset of $\mathcal{P}$ which is the image of $\mathcal{J}^1\mathcal{V}$ under $\mathbb{F}\mathcal{L}$. The coordinates we have used up to now have arisen in a natural way from coordinates on $\mathcal{M}$ and $\mathcal{V}$; they simply are the components of the tangent map of a given section. If one uses the connection Γ as a zero section of $\mathfrak{J}^1\mathcal{V} \to \mathcal{V}$ one turns $\mathfrak{J}^1\mathcal{V}$ into a vector space $\mathfrak{V}\mathcal{V} \otimes T^*\mathcal{M}$, and $\mathcal{P}$ splits into the direct sum of a line bundle and the bundle of *linear* mappings of the former vector bundle to $\Lambda^{n+1}T^*\mathcal{M}$ (cf. [17]). In coordinates this corresponds to the change

$$\Psi : (x^i, v^A, v_i^A) \mapsto (x^i, v^A, \tilde{v}_i^A = v_i^A + \Gamma_{iB}^A v^B). \tag{9}$$

Using

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial v_i^A} \circ \Psi^{-1} &= \frac{\partial \mathcal{L} \circ \Psi^{-1}}{\partial \tilde{v}_i^A}, \\ \frac{\partial \mathcal{L}}{\partial v^A} \circ \Psi^{-1} &= \frac{\partial \mathcal{L} \circ \Psi^{-1}}{\partial \tilde{v}^A} - \Gamma_{iA}^B \frac{\partial \mathcal{L} \circ \Psi^{-1}}{\partial \tilde{v}_i^B}, \end{aligned} \tag{10}$$

Eq. (2) becomes (for $\mathcal{L}_\Gamma = \mathcal{L} \circ \Psi^{-1}$)

$$\frac{\partial \mathcal{L}_\Gamma}{\partial v^A} \circ j^1\varphi - \nabla_i \left(\frac{\partial \mathcal{L}_\Gamma}{\partial \tilde{v}^A_i} \circ j^1\varphi \right) = 0. \tag{11}$$

For the affine bundle $\mathcal{P}$ the change of coordinates induces a mapping

$$\Psi^* : (x^i, v^A, p, p^{\,i}_{A}) \mapsto (x^i, v^A, p + \Gamma^A_{iB}\, p^{\,i}_{A}\, v^B, p^{\,i}_{A}).$$

Let $\mathcal{H}_\Gamma = H \circ (\Psi^*)^{-1}$. Since we have a global splitting of $\mathcal{P}$ induced by the connection Γ, this is a function on $(\mathfrak{V}\mathcal{V} \otimes T^*\mathcal{M})^*$. Differentiating $\mathcal{H}_\Gamma$ as in (8) with respect to v^A and p^i_A one obtains on solutions $j^1\varphi$ of (2)

$$\frac{\partial \mathcal{H}_\Gamma}{\partial v^A} = \nabla_i \tilde{p}^{\,i}_{A}, \qquad \frac{\partial \mathcal{H}_\Gamma}{\partial \tilde{p}^{\,i}_{A}} = -\nabla_i v^A. \tag{12}$$

For the last equation we have used that in the coordinates introduced the first jet prolongation has the form (1). A similar result can be found in [5].

Now we are going to formulate the equations of motion in a coordinate-free manner. Let the solutions of (2) be described by $(n+1)$-vector fields $\overset{n+1}{X} \in \Gamma(\Lambda^{n+1}T(\mathfrak{J}^1\mathcal{V})^*)$ with $T\bar{\pi}\overset{n+1}{X} \neq 0$. Further, let $\overset{n+1}{X}{}^V = \overset{n+1}{X} - (T\bar{\pi}\overset{n+1}{X})^h$ be the vertical component of $\overset{n+1}{X}$, where $(T\bar{\pi}\overset{n+1}{X})^h$ is the horizontal lift according to the splitting induced by the mapping (43) in Appendix B. If $\Omega^{(2,n)} = d^V\Theta^{(1,n)}$, where $\Theta^{(1,n)}$ denotes the vertical component of Θ (so that in the splitting above $\Omega^{(2,n)}$ has two vertical and n horizontal components),

$$\Theta^{(1,n)} = \Theta - \Theta^H, \qquad (X)^h \lrcorner\, \Theta^{(1,n)} = 0, \qquad \forall X \in \Lambda^{n+1}T\mathcal{M},$$
$$X \lrcorner\, \Theta^H = 0, \qquad \forall X \in \mathfrak{V}\mathcal{P},$$

the generalized Hamilton equations (8) are equivalent to

$$\left(X^V \lrcorner\, \Omega^{(2,n)}\right)^{(1,0)} = (-)^{n+1} d^V H.$$

5. Hamiltonian forms and a graded Poisson structure

With the help of the vertical exterior derivative we can define for every form Φ on $T\mathcal{P}$ the graded vertical Lie derivative by an r-vector field by

$$\mathcal{L}_{\overset{r}{X}} \Phi = \overset{r}{X} \lrcorner\, d^V\Phi + (-)^{r+1} d^V \left(\overset{r}{X} \lrcorner\, \Phi \right). \tag{13}$$

An r-vector field $\overset{r}{X}$ is called a Hamiltonian multi-vector field iff there is a horizontal $(n+1-r)$-form $\overset{(n+1-r)}{F}$ that satisfies

$$\overset{r}{X} \lrcorner\, \Omega^{(2,n)} = d^V \overset{(n+1-r)}{F}. \tag{14}$$

The set of all such forms will be called the set of Hamiltonian forms and denoted by $\mathcal{HF}$. Not every horizontal form is automatically Hamiltonian. Indeed, if we write in local coordinates

$$\overset{(n+1-r)}{F} = \frac{1}{r!} F^{i_1 \cdots i_r} (\mathfrak{e}_{i_1 \cdots i_r} \lrcorner\, \omega), \tag{15}$$

where ω is the horizontally lifted volume form of $\mathcal{M}$ and $\mathfrak{e}_{i_1 \cdots i_r} = \mathfrak{e}_{i_1} \wedge \cdots \wedge \mathfrak{e}_{i_r}$, we find for $n+1 > r$ ([12])

$$\begin{aligned} r\, X^{A[j_1 \cdots j_{r-1}} \delta_j^{i]} &= \partial_{\underset{A}{i}} F^{j_1 \cdots j_{r-1} i}, \\ -r\, X^{\overset{i}{A} j_1 \cdots j_{r-1}} &= \partial_A F^{j_1 \cdots j_{r-1} i}, \end{aligned} \tag{16}$$

which puts a restriction on the admissible horizontal forms F with $r < n+1$, namely

$$\partial_{\underset{B}{k}} F^{j_1 \cdots j_r} = 0 \tag{17}$$

for all $k \notin \{j_1, \cdots, j_r\}$. For $r = n+1$ the first equation in (16) does not lead to any restriction, since j has to be in $\{j_1, \ldots, j_n, i\}$ in any case. Moreover, from $d^V \overset{r}{X} \lrcorner\, \Omega^{(2,n)} = (d^V)^2 \overset{(n+1-r)}{F} = 0$ we deduce, in particular,

$$\sum_{i=1}^{n+1} \sum_{A,B=1}^{N} \partial_{\underset{A}{i}} X^{B i_1 \cdots i_r} \mathfrak{e}^{\overset{i}{A}} \wedge \mathfrak{e}^{\overset{j}{B}} \wedge \mathfrak{e}_{i_1 \cdots i_r j} \lrcorner\, \omega = 0,$$

which implies

$$\big(\partial_{\underset{A}{j_1}}\big)^2 F^{j_1 \cdots j_r} = -r \partial_{\underset{A}{j_1}} X^{B j_1 \cdots j_{r-1}} = 0 \quad \text{(no summation over } j_1). \tag{18}$$

Hence, as already remarked in [10], the coordinate expression of F can depend on the coordinates of the fibre of $\mathcal{P}$ in a specific polynomial way only, where each coordinate $p^{\overset{i}{A}}$ appears at most to the first power.

If $n = 0$, then $\Omega^{(2,0)}$ does not contain any horizontal degree and the Hamiltonian forms are just functions on $\mathcal{P}$. For those, the conditions (16) become

$$X^A = \partial_{\underset{A}{1}} F^1, \quad X^{\overset{1}{A}} = \partial_A F^1. \tag{19}$$

Hence, arbitrary functions F are allowed.

LEMMA 5.1. *Let* $\overset{(n+1-r)}{F} = \frac{1}{r!} F^{j_1 \cdots j_r} \mathfrak{e}_{j_1 \cdots j_r} \lrcorner\, \omega$ *be a Hamiltonian form. If* $r < n+1$, *then the coefficient functions are of the form*

$$F^{j_1 \cdots j_r}(x, v, p) = \frac{1}{r!} \sum_{k=0}^{r} p^{\overset{j_1}{A_1}} \cdots p^{\overset{j_k}{A_k}} f^{A_1 \cdots A_k j_{k+1} \cdots j_r}, \tag{20}$$

where the functions f *are antisymmetric in the upper indices.*

If $n+1=r$, then the set of Hamiltonian forms consists of all functions on the phase space $\mathcal{P}$.

With that we have the following observation.

LEMMA 5.2. *If $\overset{r}{X}$, $\overset{s}{X}$ are Hamiltonian multi-vector fields, then*

$$\overset{r}{X} \lrcorner \overset{s}{X} \lrcorner \Omega^{(2,n)} \tag{21}$$

is a Hamiltonian form.

Proof: This can be checked by a calculation using coordinates. Let us suppose that $n>0$. (The case $n=0$ is easy because there is no additional restriction on Hamiltonian forms apart from having horizontal degree zero.) Firstly, the above expression (21) is horizontal. Since $\overset{r}{X}$ and $\overset{s}{X}$ are assumed to be Hamiltonian, there are horizontal forms F and G satisfying (14), respectively. We will show that $\overset{r}{X} \lrcorner \overset{s}{X} \lrcorner \Omega^{(2,n)}$ is of the form (20),

$$\begin{aligned}\overset{r}{X} \lrcorner \overset{s}{X} \lrcorner \Omega^{(2,n)} &= \frac{1}{(r-1)!}\frac{1}{(s-1)!}\\ &\quad\times (-)^{(r-1)}\overset{r}{X}^{Mi_1\cdots i_{r-1}}\overset{s}{X}^{Nj_1\cdots j_{s-1}}\langle e_M \wedge e_N, e^A \wedge \overset{i}{e_A}\rangle(e_{i_1\cdots i_{r-1}j_1\cdots j_{s-1}i} \lrcorner \omega)\\ &= \frac{1}{(r+s-1)!}H^{i_1\cdots i_{r-1}j_1\cdots j_{s-1}i}\left(e_{i_1\cdots i_{r-1}j_1\cdots j_{s-1}i} \lrcorner \omega\right).\end{aligned}$$

Because of the special form of $\overset{r}{X}$ and $\overset{s}{X}$, according to Lemma 5.1 we find

$$\underset{A}{\partial_{i_1}} H^{i_1\cdots i_{r+s-1}} = -\underset{A}{\partial_{i_2}} H^{i_1\cdots i_{r+s-1}} \tag{22}$$

and

$$\underset{A}{\partial_i} H^{i_1\cdots i_{r+s-1}} = 0 \qquad \text{for } i \notin \{i_1, \cdots, i_{r+s-1}\}. \tag{23}$$

This shows that $\overset{r}{X} \lrcorner \overset{s}{X} \lrcorner \Omega^{(2,n)}$ fulfills the conditions derived from (16) and thus is Hamiltonian. □

Looking at Eq. (21) we can ask what the corresponding Hamiltonian multi-vector field might be. One calculates

$$\begin{aligned}d^V\left(\overset{r}{X} \lrcorner \overset{s}{X} \lrcorner \Omega^{(2,n)}\right) &= d^V\left(\overset{r}{X} \lrcorner \overset{s}{X} \lrcorner \Omega^{(2,n)}\right) + (-)^{r+1}\overset{r}{X} \lrcorner d^V\left(\overset{s}{X} \lrcorner \Omega^{(2,n)}\right)\\ &= \mathcal{L}_{\overset{r}{X}} \overset{s}{X} \lrcorner \Omega^{(2,n)}\end{aligned}$$

Since $\mathcal{L}_{\overset{r}{X}} \Omega^{(2,n)} = 0$, this looks like the Lie bracket of $\overset{r}{X}$ and $\overset{s}{X}$ being inserted in $\Omega^{(2,n)}$. Now in symplectic mechanics the Lie bracket of two (locally) Hamiltonian vector fields is the vector field associated to the Poisson bracket of the Hamiltonian functions of the former. Hence, by analogy, we define a bracket as follows,

$$\{\overset{r}{F}, \overset{s}{F}\} = (-1)^{n+1-r} \overset{n+1-r}{X} \lrcorner \overset{n+1-s}{X} \lrcorner \Omega^{(2,n)}, \tag{24}$$

where $\overset{r}{F}, \overset{s}{F}$ are Hamiltonian forms and $\overset{n-r}{X}, \overset{n-s}{X}$ denote the corresponding vector fields. Note that whereas there is some ambiguity in the choice of a Hamiltonian (multi-)vector field in Eq. (14), this does not lead to an ambiguity of the above bracket. Indeed, a vector field X that vanishes on $\Omega^{(2,n)}$ must have vanishing coefficients $X^{Mi_1 \cdots i_k}$ but can have nonvanishing components $X^{M_1 \cdots M_j i_1 \cdots i_l}$. The latter, however, do not contribute to the bracket since $\Omega^{(2,n)}$ is of type $(2, n)$[6].

PROPOSITION 5.1. *The bracket*

$$\{\cdot, \cdot\} : \mathcal{HF} \times \mathcal{HF} \rightarrow \mathcal{HF} \tag{25}$$

defined by (24) *has the following properties:*

1) *it is graded antisymmetric,*

$$\{\overset{r}{F}, \overset{s}{F}\} = -(-1)^{(n-r)(n-s)}\{\overset{s}{F}, \overset{r}{F}\}.$$

2) *it fulfills a graded Jacobi identity,*

$$(-1)^{(n-r)(n-t)}\{\overset{r}{F}, \{\overset{s}{F}, \overset{t}{F}\}\} + (-1)^{(n-s)(n-r)}\{\overset{s}{F}, \{\overset{t}{F}, \overset{r}{F}\}\} + (-1)^{(n-t)(n-s)}\{\overset{t}{F}, \{\overset{r}{F}, \overset{s}{F}\}\} = 0.$$

3) *there is a product*

$$\overset{r}{F} \bullet \overset{s}{F} = *^{-1}(*\overset{r}{F} \wedge *\overset{s}{F}) = (-1)^{(n+1-r)(n+1-s)} \overset{s}{F} \bullet \overset{r}{F}, \tag{26}$$

where $*$ is the operation induced by the Hodge operator on $\mathcal{M}$ that maps Hamiltonian functions to Hamiltonian functions. With respect to $\bullet$, the above defined bracket shows a graded Leibniz rule,

$$\{\overset{r}{F}, \overset{s}{F} \bullet \overset{t}{F}\} = \{\overset{r}{F}, \overset{s}{F}\} \bullet \overset{t}{F} + (-1)^{(n-r)(n+1-s)} \overset{s}{F} \bullet \{\overset{r}{F}, \overset{t}{F}\}. \tag{27}$$

Proof: 1) is an immediate consequence of the definition.
2) is a straightforward calculation if one uses

$$\partial_{\underset{B}{k}} X^{\overset{i}{A} j_1 \cdots j_{-1}} = -\partial_A X^{b[j_1 \cdots j_{-1}} \delta_k^{i]}, \qquad \partial_B X^{\overset{i}{A} j_1 \cdots j_{-1}} = \partial_A X^{\overset{i}{B} j_1 \cdots j_{-1}} \tag{28}$$

which can be deduced from changing the order of differentiation in (16).

6 The author wishes to thank the referees for pointing out the remaining ambiguity to him.

As for 3), using

$$*\left(e_{i_1\cdots i_r} \lrcorner e^1 \wedge \cdots \wedge e^n\right) = e^{i_1} \wedge \cdots \wedge e^{i_r}$$

we find

$$\overset{n+1-q}{G} \bullet \overset{n+1-r}{H} = \frac{1}{(q+r)!} G^{i_1\cdots i_q} H^{i_{q+1}\cdots i_{q+r}} \left(e_{i_1\cdots i_{q+r}} \lrcorner \omega\right), \tag{29}$$

and hence

$$\begin{aligned}\{\overset{n+1-p}{F}, \overset{n+1-q}{G} \bullet \overset{n+1-r}{H}\} &= (-1)^p \frac{1}{(p-1)!} X_F^{Mi_1\cdots i_{p-1}} \lrcorner d^V (G \bullet H)\\ &= X_F^{Mi_1\cdots i_{p-1}} (\partial_M G^{j_1\cdots j_q} H^{j_{q+1}\cdots j_{q+r}} e_{i_1\cdots i_{(p-1)} j_1\cdots j_{q+r}} \lrcorner \omega\\ &\quad + (-1)^{(p-1)q} G^{j_1\cdots j_q} X_F^{Mi_1\cdots i_{p-1}}\\ &\quad \times (\partial_M H^{j_{q+1}\cdots j_{q+r}}) e_{j_1\cdots j_q i_1\cdots i_{(p-1)} j_1\cdots j_{q+r}} \lrcorner \omega\\ &= \{\overset{p}{F}, \overset{q}{G}\} \bullet \overset{r}{H} + (-1)^{(p-1)q} \overset{q}{G} \bullet \{\overset{p}{F}, \overset{r}{H}\}.\end{aligned} \tag{30}$$

□

One might ask about the dependence of the bracket on the connections Γ and Λ. As can be seen from (6), different choices of connections amount to differences in the horizontal terms of the vertical forms that have been used in the definition of d^V. But from (14) we learn that this change can have an effect on those terms of X that have two or more vertical components only. Again, those terms do not contribute to the bracket. Hence the Poisson bracket does not depend on Γ nor Λ.

6. Recovering mechanics

To recover Hamiltonian mechanics we proceed as follows. Let $\mathcal{Q}$ be the coordinate space of the theory. Then, $\mathcal{M} = \mathbb{R}$ and $\mathcal{V}$ is trivial, $\mathcal{V} = \mathbb{R} \times \mathcal{Q}$. Hence, $T\mathcal{V}$ decomposes into $T\mathcal{V} = \mathbb{R} \oplus T\mathcal{Q}$. The condition for a mapping $\varphi \oplus \psi : T\mathcal{M} = \mathbb{R} \to T\mathcal{V} = \mathbb{R} \oplus T\mathcal{Q}$ to be in $\mathfrak{J}^1\mathcal{V}$ is thus

$$T\pi \circ (\varphi \oplus \psi) = \psi = T\,\mathrm{id}_{\mathbb{R}} = 1. \tag{31}$$

As the mapping φ is defined by its value at 1 we conclude $\mathfrak{J}^1\mathcal{V} = T\mathcal{Q} \times \mathbb{R}$ and going to the dual we obtain the phase space,

$$\mathcal{P}(\mathfrak{J}^1\mathcal{V})^* = (T^*\mathcal{Q} \oplus \mathbb{R}) \times \mathbb{R}. \tag{32}$$

The canonical 1-form Θ reads

$$\Theta(t, v^A, p, p_A) = p_A\, dv^A + p\, dt,$$

whereas $\Omega^{(2,0)}$ is

$$\Omega^{(2,0)}(t, v^A, p, p_A) = dp_A \wedge dv^A$$

which is just the canonical 2-form. As the base manifold is one-dimensional, horizontal forms are either functions or 1-forms on T^*Q. Now in this case Eq. (14) admits the former case since $\Omega^{(2,0)}$ does not contain any horizontal component. Therefore the Hamiltonian multi-vector fields can be ordinary vector fields on T^*Q only, and we have

$$X_F(t, v, p) = \partial_{p^A} F(t, v, p)\partial_{p^A} - \partial_{v^A} F(t, v, p)\partial_{v^A}. \tag{33}$$

There is no additional restriction to admissible Hamiltonian functions (cf. (19)) and we have arrived at the stage of Hamiltonian mechanics (cf. [9]). As the bundle $\mathcal{V}$ is trivial we do not need a connection, so there is no need for Q to be a vector bundle. As the base manifold is one-dimensional only, the product of two Hamiltonian forms always gives zero. This can be remedied if one includes horizontal 1-forms in the set of observables in addition to functions[7]. This leads to the extension of the notion of Hamiltonian vector fields to form valued vector-fields.

In [6], Section 4, where a Poisson structure is defined on (de Rham) equivalence classes of forms on $\mathcal{P}$, the Poisson algebra consists of those functions only for which the dependence on the parameter is the physical time, i.e. which solve the equations of motion when differentiated with respect to this parameter. Here, in contrast, nothing can be said about the "time" dependence of Hamiltonian forms.

7. The case of a scalar field

In the case of a scalar field, the fibre of $\mathcal{V}$ is isomorphic to $\mathbb{R}$. Using a connection $\Gamma : \mathcal{V} \to \mathfrak{J}^1\mathcal{V}$, we obtain an isomorphism

$$\mathfrak{J}^1\mathcal{V} \overset{\Gamma}{\cong} \mathfrak{V}\mathcal{V} \otimes_{\mathcal{V}} T^*\mathcal{M}, \quad \mathfrak{V}\mathcal{V} \cong \mathbb{R} \times \mathbb{R}. \tag{34}$$

Hence

$$\mathfrak{J}^1\mathcal{V} \overset{\Gamma}{\cong} \mathrm{pr}^*(T^*\mathcal{M}), \tag{35}$$

where pr denotes the canonical projection of the bundle $\mathcal{V} \to \mathcal{M}$. Using (14) one immediately verifies in local coordinates (x^i, v, p^i, p) of $\mathcal{P}$ in this case (let e_i denote the horizontal lifts of tangent vectors of $\mathcal{M}$ and ϵ^i be the vertical forms with respect to the splitting discussed in Appendix; the determinant comes from the volume element on $\mathcal{M}$)

$$-\partial_v \lrcorner\, \Omega^{(2,n)} = \epsilon^i \wedge (e_i \lrcorner\, \omega) = d^V p^i \wedge (e_i \lrcorner\, \omega),$$

$$\sum_{i=1}^{n+1} \partial_{p^i} \wedge ((-1)^i(\sqrt{\det g})e_1 \wedge \cdots \wedge e_{i-1} \wedge e_{i+1} \wedge \cdots \wedge e_{n+1}) \lrcorner\, \Omega^{(2,n)} = d^V v,$$

hence $\Pi(x, v, p) = p^i \wedge (e_i \lrcorner\, \omega)$ satisfies

$$\{\Pi, \Phi\} = 1$$

[7] I. Kanatchikov, private communication.

for $\Phi(x, v) = v$, but $\Pi \bullet 1 = 0 = \Phi \bullet 1$. The unit with respect to $\bullet$ is ω, so one should look for solutions of

$$X \lrcorner Y \lrcorner \Omega^{(2,n)} = \omega.$$

This cannot be solved, as $\Omega^{(2,n)}$ contains n horizontal components, whereas ω is a horizontal $(n+1)$-form. As before, one might have to include vector fields that are form valued, i.e. endomorphisms of $\Lambda^* T^* \mathcal{P}$. Note, however, that the connection Γ remains arbitrary; athough it is hidden in the expression for Π,

$$\Pi(x, v, p) = p^i \wedge (e_i \lrcorner \omega) = p^i \wedge (\partial_i \lrcorner \omega),$$

Π is in fact independent of it.

8. Conclusions

In multisymplectic geometry we take the phase space $\mathcal{P}$ to be the affine dual of the first jet bundle to a given vector bundle $\mathcal{V}$. It is then possible to define (graded) Poisson brackets (24) on $\mathcal{P}$ even for nontrivial vector bundles. For this one needs a covariant derivative on the $(n+1)$-dimensional base manifold $\mathcal{M}$(space-time) and a connection on the vector bundle of the fields under consideration.

Kanatchikov has proposed a similar construction by making use of equivalence classes of forms modulo forms of higher horizontal degree [11]. This is equivalent to he use of the construction elaborated in this article, as a vertical form, say ϵ^A, differs from the coordinate expression dv^A by horizontal components only, cf. (6),

$$\epsilon^{*A}_{(p)} = dv^A + \Gamma^A_{iB}(\pi(p))\, v^B\, dx^i. \tag{36}$$

Hence, $\epsilon^{*A}_{(p)}$ and dv^A define the same equivalence class, independent of the connections Γ and Λ used. The same applies to the bracket. Whereas the correspondence of Hamiltonian forms and multi-vector fields is ambiguous and depend on the connections chosen, the (graded) Poisson bracket does not. Admissible observables are the alled Hamiltonian forms, horizontal forms that satisfy certain consistency relations (16). It turns out that those Hamiltonian forms are polynomial in the momenta, i.e. coordinates of the fibres of $\mathcal{P} \to \mathcal{V}$, cf. (20).

In addition $\mathcal{M}$ has to be orientable in order to define the multiplication (26) between Hamiltonian forms. For Hamiltonian forms of the same degree, this product is commutative but gives zero if the form degree is less than $(n+1)/2$.

If space-time is taken to be one-dimensional the whole formalism reduces to ordinary mechanics on a configuration space $\mathcal{Q}$. Hamiltonian forms then are arbitrary functions on the extended phase space $T^*\mathcal{Q} \times \mathbb{R}$, and the Poisson bracket takes the standard form. However, the product $\bullet$ of functions always gives zero in this case.

In the case of a scalar field, given a (local) field Φ one can define a Hamiltonian form Π that satisfies $\{\Pi, \Phi\} = 1$, but the constant function 1 is not the unit with respect to $\bullet$. Rather, this rôle is played by ω, the pulled back volume form from $\mathcal{M}$. To obtain $\{\Pi, \Phi\} = \omega$ one has to extend the notion of Hamiltonian vector fields in a way similar to that needed in the mechanical case (as mentioned above), namely one has to include form-valued vector fields, i.e. endomorphisms of $\Lambda^* T^* \mathcal{P}$.

The Poisson structure is graded in the following way. Let the degree of a (homogeneous) Hamiltonian form be its degree as an element of the exterior algebra. Then the degree of the Poisson bracket of two Hamiltonian forms is the sum of the respective degree minus n, the number of space directions, while the degree of the product of two Hamiltonian forms is the sum of the degrees minus $n+1$,

$$\deg\{\overset{r}{F}, \overset{s}{F}\} = \deg \overset{r}{F} + \deg \overset{s}{F} - n, \qquad \deg \overset{r}{F} \bullet \overset{s}{F} = \deg \overset{r}{F} + \deg \overset{s}{F} - (n+1). \quad (37)$$

Looking at Proposition 5.1 we find that the graded antisymmetry of the $\{,\}$, the graded Jacobi identity, the graded derivation property with respect to $\bullet$ and the graded commutativity of $\bullet$ all match with each other.

As already remarked in the examples, how to relate observables of physical fields and Hamiltonian forms? This point requires further investigation, especially the relation with the multiplicative structure. In particular, the notion of canonical conjugate momenta needs to be clarified.

Note added. As pointed out by one of the referees the above construction depends heavily on the vector space structure of fibres of $\mathcal{V}$. This might be sufficient for the study of such field theories where the fields take their values in a vector space. For classical mechanics on arbitrary configurations spaces, or in the case of string theory—whenever the target space is not Minkowski space—there is indeed a need for a generalization of the construction. In this article, all that is used really is a splitting of the tangent space $T\mathfrak{J}^1\mathcal{V}$ in horizontal and vertical subspaces with respect to the canonical projection onto $\mathcal{M}$. Such a splitting does not exist canonically. There is, however, a natural way to split $(\pi_1)_0^{1*}\left(T\mathfrak{J}^1\mathcal{V}\right)$, the pull back of $T\mathfrak{J}^1\mathcal{V}$ onto $\mathfrak{J}^1\mathfrak{J}^1\mathcal{V}$, the first jet bundle of $\mathfrak{J}^1\mathcal{V}$. Now every connection $\bar{\Gamma}$ on $\mathfrak{J}^1\mathcal{V}$ (viewed as a bundle over $\mathcal{M}$) defines a map $\bar{\Gamma} : \mathfrak{J}^1\mathcal{V} \to \mathfrak{J}^1\mathfrak{J}^1\mathcal{V}$ and hence induces a splitting of $T\mathfrak{J}^1\mathcal{V}$. For $\mathcal{V}$ being a general fibre bundle, the connection Γ does not depend linearly on the fibre coordinates (cf. (39)). Rather, it takes the most general form

$$\Gamma : \mathcal{V} \ni (x^i, u^A) \mapsto (x^i, u^A, \Gamma_i^A).$$

In this case, in the local expression (42), one has to replace $-\Gamma_{iB}^A u_j^B$ by $\partial_{u^B}\Gamma_i^A u_j^B$ and $\Gamma_{kB}^A u^B$ by $-\Gamma_k^A$.

Acknowledgements

The author's interest in this subject was initiated by very elucidating discussions with H. Römer and M. Bordemann about quantization schemes for field theories. In particular, the author thanks M. Bordemann for explaining [2] to him and for critical remarks. Finally, clarifying discussions with and valuable comments by I. Kanatchikov are gratefully acknowledged.

A. Connections and jet bundles

Given a bundle $\pi : \mathcal{V} \to \mathcal{M}$ over an n-dimensional base manifold $\mathcal{M}$ every connection is defined by a section Γ of the first jet bundle $\mathfrak{J}^1\mathcal{V}$ of $\mathcal{V}$, since it describes how to lift tangent vectors of the base manifold horizontally. If in addition $\mathcal{V}$ is a vector bundle (with fibre V) then as $\mathfrak{J}^1\mathcal{V}$ is an affine bundle over $\mathcal{V}$ the connection Γ delivers an isomorphism

$$\mathfrak{J}^1\mathcal{V} \overset{\Gamma}{\cong} \mathfrak{V}\mathcal{V} \otimes_{\mathcal{M}} T^*\mathcal{M}, \tag{38}$$

where both sides ($\mathfrak{V}\mathcal{V}$ being the vertical bundle to $\mathcal{V}$) are viewed as bundles over the base manifold $\mathcal{M}$. Note in particular that the vertical bundle $\mathfrak{V}\mathcal{V}$ is a vector bundle over $\mathcal{M}$ (with typical fibre $V \times V$, [14], Chapter II, 6.11.).

Now for $\mathcal{V}$ being a vector bundle we can form the covariant derivative ∇ that corresponds to the given connection Γ. Then horizontal lifts of tangent vectors are represented by covariantly constant lifts of curves in the base manifold $\mathcal{M}$. Therefore, in local coordinates $(x^i)_{i=1,\dots,n}$ of $\mathcal{M}$ and $(x^i, v^A)_{i=1,\dots,n,A=1,\dots,N}$ of $\mathcal{V}$ the map $\Gamma(v) \in \left(\mathfrak{J}^1\mathcal{V}\right)_v$, $v \in \mathcal{V}$, takes the form

$$\Gamma(v) : \; (x, \dot{c}^i(x)) \mapsto \left(x, v, -\Gamma^A_{iB}(x)\, v^B\right), \tag{39}$$

where $\Gamma^A_{iB}(x)$ is the Christoffel symbol of ∇.

Now we are looking for a connection in $\mathfrak{J}^1\mathcal{V}$, that is for a map

$$\bar{\Gamma} : \mathfrak{J}^1\mathcal{V} \to \mathfrak{J}^1\left(\mathfrak{J}^1\mathcal{V}\right).$$

For this, one needs a connection both in $\mathcal{V}$ and $\mathcal{M}$ ([13], Proposition 4). If we use the isomorphisms

$$\mathfrak{J}^1\mathcal{V} \overset{\Gamma}{\cong} \mathfrak{V}\mathcal{V} \otimes T^*\mathcal{M} \quad \text{and} \quad \mathfrak{J}^1\left(\mathfrak{V}\mathcal{V} \otimes T^*\mathcal{M}\right) \cong \mathfrak{J}^1\left(\mathfrak{V}\mathcal{V}\right) \otimes \mathfrak{J}^1\left(T^*\mathcal{M}\right),$$

the latter being natural, we see that all we need is a map $\mathfrak{V}\mathcal{V} \to \mathfrak{J}^1\mathfrak{V}\mathcal{V}$, since a connection on $\mathcal{M}$ defines a map $\Lambda^* : T^*\mathcal{M} \to \mathfrak{J}^1\left(T^*\mathcal{M}\right)$. Now the desired map can be constructed by vertical prolongation if we make use of the isomorphism $\mathfrak{V}\mathfrak{J}^1\mathcal{V} \cong \mathfrak{J}^1\mathfrak{V}\mathcal{V}$ ([6], Eq. (1.4))[8]:

$$\mathfrak{V}\Gamma : \mathfrak{V}\mathcal{V} \to \mathfrak{V}\mathfrak{J}^1\mathcal{V} \cong \mathfrak{J}^1\mathfrak{V}\mathcal{V}.$$

Indeed,

$$\mathfrak{V}\Gamma \otimes \Lambda^* : \mathfrak{V}\mathcal{V} \otimes T^*\mathcal{M} \to \mathfrak{J}^1\mathfrak{V}\mathcal{V} \otimes \mathfrak{J}^1 T^*\mathcal{M} \cong \mathfrak{J}^1(\mathfrak{V}\mathcal{V} \otimes T^*\mathcal{M})$$

[8] Let s_t denote a one-parameter family of local sections of $\pi : \mathcal{V} \to \mathcal{M}$. Then

$$\frac{d}{dt}\Big|_{t=0} j^1(s_t)(x) \mapsto j^1\left(\frac{d}{dt}\Big|_{t=0} s_t\right)(x)$$

gives the isomorphism.

gives a connection[9]

$$\bar{\Gamma} : \mathfrak{J}^1\mathcal{V} \to \mathfrak{J}^1\left(\mathfrak{J}^1\mathcal{V}\right). \tag{40}$$

In coordinates (x^i, v^A, v_i^A) of $\mathfrak{J}^1\mathcal{V}$ one calculates

$$\bar{\Gamma}(x^i, v^A, v_i^A) : \left(x^i, \dot{x}^i\right) \mapsto \left(x^i, v^A, v_i^A, \dot{x}^i, -\Gamma^A_{jB}(x)\, v^B \dot{x}^j, \bar{\Gamma}^A_{ij}\dot{x}^j\right), \tag{41}$$

where

$$\bar{\Gamma}^A_{ij}(x^i, u^A, u^A_i) = -\Gamma^A_{jB}(u^B_i + \Gamma^B_{iC}u^C) - \Lambda^k_{ji}(u^A_k + \Gamma^A_{kB}u^B) - (\partial_j\Gamma^A_{iB})u^B + \Gamma^A_{iB}\Gamma^B_{jC}u^C. \tag{42}$$

Note that Λ^k_{ij} denote the Christoffel symbols of Λ, not of Λ^*.

B. A covariant derivative on $T\mathcal{P}$

Using a connection Γ of $\pi : \mathcal{V} \to \mathcal{M}$, which is a map

$$\Gamma : \mathcal{V} \to \mathfrak{J}^1\mathcal{V},$$

the affine bundle $\pi' : \mathfrak{J}^1\mathcal{V} \to \mathcal{V}$ becomes a vector bundle,

$$\mathfrak{J}^1\mathcal{V} \overset{\Gamma}{\cong} \mathfrak{V}\mathcal{V} \otimes_{\mathcal{V}} \pi^*\left(T^*\mathcal{M}\right),$$

where $\Gamma(\mathcal{V})$ is identified with the zero section.

If in addition π is a vector bundle, then $\mathfrak{V}\mathcal{V}$ is a vector bundle over $\mathcal{M}$ as well ([14], Chapter II, 6.11), and we have

$$\mathfrak{J}^1\mathcal{V} \overset{\Gamma}{\cong} \mathfrak{V}\mathcal{V} \otimes_{\mathcal{M}} T^*\mathcal{M}.$$

Let $\overline{\mathcal{V}} = \mathfrak{V}\mathcal{V} \otimes_{\mathcal{M}} T^*\mathcal{M}$. In multisymplectic geometry, the phase space $(\mathfrak{J}^1\mathcal{V})^*$ consists of all with respect to π' fibre-wise affine mappings from $\mathfrak{J}^1\mathcal{V}$ to $\Lambda^n T^*\mathcal{M}$. In order to simplify the notation, let us denote this bundle by $\mathcal{P} := (\mathfrak{J}^1\mathcal{V})^*$. Again, the connection Γ provides an isomorphism

$$\mathcal{P} \overset{\Gamma}{\cong} (\overline{\mathcal{V}}^* \otimes \Lambda^n T^*\mathcal{M}) \oplus_{\mathcal{V}} \mathbb{R},$$

where $\tilde{p} \in \mathcal{P}$ is decomposed into a linear map $\bar{p} : \overline{\mathcal{V}} \to \Lambda^n T^*\mathcal{M}$ and a function p on $\mathcal{V}$ in the following way,

$$\tilde{p}(\tilde{v}) = \tilde{p}(\tilde{v}) - \tilde{p}(\Gamma(\pi'(\tilde{v}))) + \tilde{p}(\Gamma(\pi'(\tilde{v}))) = \bar{p}(\bar{v}) + p(v).$$

Making use of the duality of $\overline{\mathcal{V}}^*$ and $\overline{\mathcal{V}}$, we obtain a connection $\overline{\Gamma}^*$ on $\overline{\mathcal{V}}^*$ by

$$\langle \overline{\Gamma}^*(v), \bar{v}\rangle = \langle \bar{p}, \overline{\Gamma}(v)\rangle, \quad \forall v \in \mathcal{V}, \bar{v} \in \overline{\mathcal{V}}_v, \bar{p} \in \overline{\mathcal{V}}^*_v.$$

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[9] In [13], p. 136, this construction is denoted by $p(\Gamma, \Lambda)$.

Here, $\overline{\Gamma}$ is the connection on $\overline{\mathcal{V}}$ as explained in detail in (A). Further, this gives a connection on $\mathcal{P}$. In coordinates (x^i, v^A, p^i_A, p) we calculate

$$\overline{\Gamma}^*(\bar{p}) : T_x\mathcal{M} \ni (x^i, \xi^i) \mapsto (x^i, -\Gamma^A_{iB}\, v^B\, \xi^i, (\Lambda^k_{ji} p^i_A - \Gamma^B_{jA} p^k_B)\xi^j, 0) \in T\mathcal{P}.$$

Now $\overline{\Gamma}^*$ defines a covariant derivative $\bar{\nabla}$ on $\mathcal{P}$. With the help of this we define the connection mapping K for $[\alpha]_p] \in T_p\mathcal{P}$, represented by a curve $\alpha(t)$, by

$$K : T_p\mathcal{P} \ni [\alpha]_p \mapsto \begin{cases} \frac{d}{dt}\,|_{t=0}\, \alpha(t) & \text{if } T\bar{\pi}[\alpha] = 0, \\ \left(\bar{\nabla}_{T\bar{\pi}[\alpha]}\alpha\right)(0) & \text{otherwise.} \end{cases} \tag{43}$$

One easily verifies that K is well defined. Let p be a point in $\mathcal{P}$ and x its image under the projection $\bar{\pi}$. For the tangent mapping of the canonical projection $\bar{\pi} : \mathcal{P} \to \mathcal{M}$, the map $K \oplus T\bar{\pi} : T_p\mathcal{P} \to \mathcal{P}_x \oplus T_x\mathcal{M}$ is bijective and hence provides a splitting of $T\mathcal{P}_p$. $X^h_p \in T_p\mathcal{P}$ is called the horizontal lift of $H \in T_x\mathcal{M}$ iff $K \oplus T\bar{\pi}(X^h_p) = X$. Similarly, $q^v_p \in T_p\mathcal{P}$ is called the vertical lift of $q \in \mathcal{P}_x$ iff $K \oplus T\bar{\pi}(q^v_p) = q$. Using this we define a covariant derivative D on $T\mathcal{P}$ by[10]:

$$\begin{aligned} D_{X^h}Y^h\big|_p &= \left(\nabla^{\mathcal{M}}_X Y\right)^h\big|_p + \frac{1}{2}\left(\bar{R}(X,Y)p\right)^v\big|_p \\ D_{X^h}\beta^v\big|_p &= \left(\bar{\nabla}_X\beta\right)^v\big|_p \\ D_{\beta^v}X^h\big|_p &= 0 = D_{\beta^v}\Gamma^v\big|_p, \end{aligned} \tag{44}$$

where $p \in \mathcal{P}$, $\beta^v, \Gamma^v, X^h, Y^h \in T\mathcal{P}$ are lifts as above, and $\nabla^{\mathcal{M}}$ is the (torsion free) covariant derivative on $T\mathcal{M}$. The curvature term $\bar{R}$ of $\bar{\nabla}$ is needed for D to be torsion free.

Since at every point p of $\mathcal{P}$ the tangent space $T_p\mathcal{P}$ decomposes into the direct sum of horizontal and vertical vectors, we can choose an appropriate basis as follows. If (x^i) are coordinates of a neighbourhood $\mathcal{U}$ of $\mathcal{M}$ that trivialises $\mathcal{P}|\mathcal{U}$ and $(\xi^i, v^A, \overset{i}{p}_A, p)$ are coordinates on $\mathcal{P}$, we define for every $p \in \mathcal{P}$

$$\begin{aligned} &\mathfrak{e}_i(p) = *\,(\partial_{x^i})^h\big|_p = \partial_{\xi^i} - \Gamma^B_{iA}\, v^A\, \partial_{v^B} + \overline{\Gamma}^A_{ij}\, \partial_{\overset{j}{p}_A} \\ &\mathfrak{e}_A(p) = \partial_{v^A}, \quad \mathfrak{e}_{\underset{A}{i}}(p) = \partial_{\overset{i}{p}_A}, \quad \mathfrak{e}(p) = \partial_p, \quad i = 1, \ldots, n, \;\; A = 1, \ldots, N, \end{aligned}$$

and we obtain a basis of $T_p\mathcal{P}$. From the definition of D it follows in particular that

$$D_{\mathfrak{e}_A}\mathfrak{e}_\alpha = 0, \qquad D_{\mathfrak{e}_{\underset{A}{i}}}\mathfrak{e}_\alpha = 0, \qquad \forall \alpha = i, A, \overset{j}{_B}, \quad A, B = 1, \ldots, N, \quad i, j = 1, \ldots, n.$$

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[10] This method is inspired by the construction in [2].

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