Yang-Mills Action from Minimally Coupled Bosons

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Results

Regularized determinants

$$\mathrm{tr}_{\Lambda}\left\{\log(-\Delta_A+m^2)-\log(-\Delta_0+m^2)\right\}=c_2\Lambda^2+c_1\Lambda+c_{\log}\log\Lambda+\text{finite}$$

$\Lambda\ldots$momentum cut off regulator,

$\Delta_A\ldots$generalized Laplace operator:

$$\Delta_A=\sum_{\mu=1}^{4}D_{A,\mu}D_{A,\mu},\quad D_{A,\mu}\psi=\partial_\mu\psi+\mathrm{i}A_\mu*\psi,\quad \psi\in\mathbb{C}^N\otimes L^2(\mathbb{R}^4)$$

$*\ldots$ Moyal product on $\mathbb{R}^4$

- We find

$$c_{\log}=\frac{1}{96\pi^2}\int_{\mathbb{R}^4}d^4x\,\mathrm{tr}_N\left\{F^{\mu\nu}*F_{\mu\nu}\right\}$$

$F_{\mu\nu}\ldots$ generalized Yang-Mills curvature

- Can do minimal coupling to metric tensor

Motivation

- Determinants of differential operators occur in QFT on one-loop level
- Need to regularize
  1. Use $\log\det B = \operatorname{tr}\log B$
  2. Restrict trace: $\operatorname{tr}_\Lambda B = \operatorname{tr}\left\{\theta(\Lambda^2 + \Delta_0)B\right\}$
- Asymptotic expansion in Λ gives $c_{\log}$
  - Important for renormalization
  - Wodzicki residue in mathematics
- We consider bosons in minimal coupling to an external Yang-Mills field:

$$\operatorname{tr}_\Lambda\left\{\log(-\Delta_A + m^2) - \log(-\Delta_0 + m^2)\right\}$$

The Moyal Plane

- Simple example of a noncommutative geometry
- New product for functions on $\mathbb{R}^4$ with

$$[x^\mu, x^\nu]_* = \mathrm{i}\Theta^{\mu\nu},$$

$\Theta \ldots 4\times 4$ symplectic matrix $\times$ deformation parameter ϑ

- Uncertainty relation for position measurements
- Integral formula

$$(f * g)(x) = \frac{1}{(2\pi)^4}\int d^4y\, d^4\xi\, e^{\mathrm{i}\xi(x-y)}\, f(x - \tfrac{1}{2}\Theta\xi)\, g(y)$$

- $f(x) \sim |x|^{-p}$, $g(x) \sim |x|^{-q} \Rightarrow f * g(x) \sim |x|^{-(p+q)}$

Tool: Pseudodifferential Operators

- Fourier transform maps differential operator D to its symbol $\sigma[D]$:

$$\langle x|D|p\rangle = \sigma[D](x,p)\,\langle x|p\rangle$$

- Integral formula $\quad (D\psi)(x) = \frac{1}{(2\pi)^4}\int d^4y\,d^4p\,e^{\mathrm{i}p(x-y)}\,\sigma[D](x,p)\,\psi(y)$

- Allow symbols with asymptotic expansion for $m \in \mathbb{R} \Rightarrow$ pseudodifferential operator (PsDO) (“power counting”)

$$\sigma(x,p) = \sigma_m(x,p) + \sigma_{m-1}(x,p) + \ldots, \quad \sigma_k(x,\lambda p) = \lambda^k \sigma_k(x,p)$$

- Can study functions of PsDO’s

- Example: $f* : \psi \mapsto f * \psi$

$$\sigma[f*](x,p) = f(x - \tfrac{1}{2}\Theta p)$$

Note that $m = -\infty$ if f rapidly decaying

Pseudodifferential Operators: Trace

- Trace of a pseudodifferential operator D

$$\mathrm{tr}\{D\} = \int d^4p \, \langle p|D|p\rangle$$

$$= \int d^4x d^4p \, \langle p|x\rangle \langle x|D|p\rangle$$

$$= \frac{1}{(2\pi)^4} \int d^4x d^4p \, \sigma[D](x,p)$$

- In general, PsDO are not trace-class $\Rightarrow$ introduce momentum cut-off Λ for p-integration

- Wodzicki residue

$$Res D = \frac{1}{(2\pi)^4} \int_{|p|=1} d^3\Omega_p \, d^4x \sigma_{-4}[D](x,p)$$

gives precisely the factor of $\log\Lambda$ in the cut off trace

The symbol of $\log\{-\Delta_A + m^2\}$

- We use

$$\log(1+D) = \int_0^1 \frac{ds}{s}\left(1 - \frac{1}{1+sD}\right)$$

$\Rightarrow$ need resolvent $R_\lambda = (-\Delta_A + \lambda)^{-1}$

- $\sigma[R_\lambda]$ satisfies recursion relation

$$\sigma[R_\lambda](x,p) = \frac{1}{p^2+\lambda} + \frac{1}{p^2+\lambda}\left(\Delta_{\tilde{A}} + 2\mathrm{i}\, p^\mu D_{\tilde{A},\mu}\right)\sigma[R_\lambda](x,p)$$

with shifted $\tilde{A}(x,p) = A(x - \frac{1}{2}\Theta p)$

- Gives series for $\sigma[\log\{-\Delta_A + m^2\}]$ after s-integration
- Large p behaviour: two contributions
- Which terms contribute to divergent part of tr_Λ?

Problems with the asymptotic expansion

- Term no 2 in the series for $\sigma[R_\lambda]$:

  $$\frac{1}{p^2+\lambda}\Big(\mathrm{i}(\partial^\mu A_\mu)(x-\tfrac{1}{2}\Theta p)-(A^\mu * A_\mu)(x-\tfrac{1}{2}\Theta p)-2\,p^\mu A_\mu(x-\tfrac{1}{2}\Theta p)\Big)$$

  Large p behaviour depends on properties of A_μ!

- But: Change of variables $x \mapsto x+\frac{1}{2}\Theta p$ in the trace integral kills the p-dependence in the Yang-Mills fields

- Lesson: On noncompact manifolds, there may be contributions to the divergent part of the trace integral from the tail

- Example

  $$\sigma[T](x,p) = \exp\{-x^2 e^{-p^2} - \tfrac{1}{4}p^2\} \sim e^{-\frac{1}{4}p^2}$$

  $$\int dx\, \sigma[T](x,p) = \sqrt{\pi} e^{+\frac{1}{4}p^2}$$

A bound on the tail

- Remainder in series for $\sigma[R_\lambda]$ at order r is $\frac{1}{(p^2+\lambda)^{r+1}}\left(\Delta_{\tilde{A}} + 2\mathrm{i}p^\mu D_{\tilde{A},\mu}\right)^r \sigma[R_\lambda]$

  Insufficient control over $*$-action of A_μ's on $\sigma[R_\lambda]$

- Idea: Consider $R_\lambda^{(r)}$ with $\sigma[R_\lambda^{(r)}] = \sum_{k=1}^r \frac{1}{(p^2+\lambda)^{r+1}}\left(\Delta_{\tilde{A}} + 2\mathrm{i}p^\mu D_{\tilde{A},\mu}\right)^r 1$ and show that $(-\Delta_A+\lambda)R_\lambda^{(r)} = 1 + \text{ trace class }$ for large r, because

$$\frac{1}{(p^2+\lambda)^{r+1}}(A * \overset{s}{\ldots} * A)(x - \tfrac{1}{2}\Theta p)$$

  is x-p-integrable for r, s large.

- Then $(-\Delta_A+\lambda)(R_\lambda - R_\lambda^{(r)}) = \text{ trace class }$ and, since R_λ is bounded (for our values of λ),

$$R_\lambda - R_\lambda^{(r)} = \text{ trace class}$$

Calculation of $c_{\log}$

- Truncate series, perform s-integral

$$\sigma[\log(-\Delta_A + m^2)](x,p) = \log(p^2 + m^2) + \sum_{k=1}^{r} \frac{1}{k(p^2+m^2)^k} (\Delta_{\tilde{A}} + 2\mathrm{i} D_{\tilde{A},\mu})^k 1$$

- Asymptotic series of the trace, in 4D

$$\mathrm{tr}_\Lambda\{\log(-\Delta_A + m^2) - \log(-\Delta_0 + m^2)\} = c_2\Lambda^2 + c_1\Lambda + c_{\log}\log\Lambda + \dots$$

- The coefficient $c_{\log}$ is given by

$$c_{\log} = \frac{1}{96\pi^2} \int d^4x \, \mathrm{tr}_N\{F^{\mu\nu} * F_{\mu\nu}\}$$

where

$$F_{\mu\nu} = -\mathrm{i}[D_{A,\mu}, D_{A,\nu}] = \partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}[A_\mu, A_\nu]_*$$

Lessons

- Pseudodifferential operator methods provide powerful tool
- Cut off regularized trace has asymptotic expansion
- Logarithmically divergent term is gauge invariant
- Commutative case ($\vartheta = 0$): Wodzicki residue
- Useful formula $\sigma[f(-\Delta_A)](x,p) = f(p^2 - \Delta_{\tilde{A}} - 2\mathrm{i}p^\mu D_{\tilde{A},\mu})1$, where $\tilde{A}(x,p) = A(x - \frac{1}{2}\Theta p)$
- Problems with non-compactnes of Moyal plane can be overcome by adaption of pseudodifferential operator calculus, expand

$$\sigma(x + \tfrac{1}{2}\Theta p, p)$$

Curved background

- Space is $\mathbb{R}^4$ with non-constant metric tensor $g_{\mu\nu}$

- Commutative case $\theta = 0$

- Minimal coupling to metric tensor

$$\Delta_A^g = \frac{1}{\sqrt{g}} D_{A,\mu} \sqrt{g} g^{\mu\nu} D_{A,\nu}$$

- Results from zeta-function regularization exist

The trace formula on curved background

- Scalar product on curved background

$$\langle \psi, \varphi \rangle = \int \sqrt{g}\, d^4x\, \overline{\psi}(x)\varphi(x)$$

- Orthogonality and completeness relations for $\langle x|p\rangle_g = \frac{1}{\sqrt[4]{g}}\langle x|p\rangle$

$$\int \sqrt{g} d^4x_g \langle p|x\rangle \langle x|q\rangle_g = \delta^4(p-q), \quad \int d^4p_g \langle p|x\rangle \langle y|p\rangle_g = \frac{1}{\sqrt{g}}\delta^4(x-y)$$

- Trace formula

$$\begin{aligned} \mathrm{tr}D &= \int d^4p_g \langle p|D|p\rangle_g = \int d^4p \sqrt{g}\, d^4x_g \langle p|x\rangle \langle x|D|p\rangle_g \\ &= \int d^4p\, d^4x \langle p|x\rangle \langle x|D|p\rangle + \int d^4p\, d^4x \langle p|x\rangle \langle x| \sqrt[4]{g}\,[D, \frac{1}{\sqrt[4]{g}}]|p\rangle \end{aligned}$$

$c_{\log}$ on curved background

- Program as before: recursion relation for $(-\Delta_A^g + \lambda)^{-1}$
- New terms with derivatives of g arise in the recursion
- Problem: covariant derivatives D_μ do not commute with p^2
- Calculations involved, no simple formula in the end
- Simplification by use of Riemann normal coordinates to fourth order

$$g_{\mu\nu}(x+\delta) = \eta_{\mu\nu} + \frac{1}{3} R_{\mu\rho\nu\sigma} \delta^\rho \delta^\sigma + \ldots$$

- Commutator term does not contribute (no partial integration needed)

Result and Discussion

- Result without reference operator

$$c_{\log} = \frac{1}{96\pi^2} \int \sqrt{g}\, d^4x \,\Big(\mathrm{tr}_N\{F^{\mu\nu}F_{\mu\nu}\} \\ + 2N m^2 R - \frac{N}{30}(2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2Ric_{\mu\nu}Ric^{\mu\nu} + 5R^2) - N\frac{4}{3}\Delta R\Big)$$

- ΔR term absent in heat kernel calculations — asymptotical flatness?
- As a consequence of the result, the $\log\Lambda$ term in the trace of

$$\log(-\Delta_A^g + m^2) - \log(-\Delta_0^g + m^2)$$

  gives the right generalization of $c_{\log}$ from the flat case
- Why is there a separation of A and g contributions?
- Moyal plane?