

ON THE GEOMETRY OF FIELD-THEORETIC GERSTENHABER STRUCTURES

CORNELIUS PAUFLER

Fakultät für Physik, Albert-Ludwigs-Universität Freiburg im Breisgau,
Hermann-Herder-Straße 3, D-79104 Freiburg i. Br., Germany
(e-mail: paufler@physik.uni-freiburg.de)

(Received August 28, 2000 — Revised April 9, 2001)

Field-theoretical models with first order Lagrangian can be formulated in a covariant Hamiltonian formalism. In this article, the geometrical construction of the Gerstenhaber structure that encodes the equations of motion is explained for arbitrary fibre bundles. Special emphasis has been put on naturality of the constructions. Further, the treatment of symmetries is explained. Finally, the canonical field theoretical 2-form is obtained by pull-back and integration of the polysymplectic form over space like hypersurfaces.

Keywords: geometric field theory, multisymplectic geometry, Hamiltonian formulation, Gerstenhaber algebra.

1. Introduction

It has turned out that field theories can be best described in the jet bundle formalism. For first-order theories, i.e. when the Lagrange density is a function of the fields and their first derivatives only, the description can be reduced to the first jet bundle. This description is an extension of the Lagrangian formulation of mechanics in the following sense. In particle mechanics, particles are described by curves in the tangent space TQ , i.e. by sections of the (trivial) bundle $\mathbb{R} \times TQ$. In field theory, physical fields are sections of some bundle over space-time, i.e. higher dimensional curves. Consequently, the base manifold of a given field theory—which quite often is space-time—*should not be confused with the configuration space of mechanics*. Rather, the latter is represented by the fibres of the given fibre bundle when the base manifold is \mathbb{R} , i.e. the time axis.

The corresponding field equations are of second order but—for nicely behaving Lagrange densities—can be reduced to a system of first-order equations, the so-called de Donder–Weyl equations (for a review see [3]). One might view these equations as a generalisation of the usual Hamilton equations in mechanics. However, a peculiar feature of this approach is that there are as many conjugate momenta corresponding to each field coordinate degree of freedom as there are space-time dimensions.

For the study of quantum effects in field theory, at least in the sense of star products

[2] there is the need of an algebraic formulation of the theory. There has been recent progress on that, namely the Gerstenhaber structure found by Kanatchikov [4, 5].

In this article, a geometric construction of this Gerstenhaber structure will be proposed that works for arbitrary fibre bundles. Particular attention will be paid to the treatment of internal symmetries and to the correspondence to the standard Hamiltonian formulation of field theory.

2. Phase space for first order field theories

Let $(\mathcal{E}, \pi, \mathcal{M})$ be a fibre bundle with N -dimensional fibres over an n dimensional orientable manifold \mathcal{M} . Its first jet bundle, $\mathfrak{J}^1\mathcal{E}$ (cf. [9]), consists of equivalence classes of local sections of \mathcal{E} : Two sections φ and ψ are considered equivalent at a point $m \in \mathcal{M}$, if they take the same value in m , $\varphi(m) = \psi(m)$, and their tangent mappings coincide, $T_m\varphi = T_m\psi$. The class defined by a section φ is denoted by $j_m^1\varphi$. Let π_1 and π_0^1 be the respective projections of $j_m^1\varphi$ onto m and $\varphi(m)$. Then $(\mathfrak{J}^1\mathcal{E}, \pi_1, \mathcal{M})$ is a fibre bundle of fibre dimension $(Nn + N + n)$, while $(\mathfrak{J}^1\mathcal{E}, \pi_0^1, \mathcal{E})$ is an affine bundle with associated vector bundle $(\mathfrak{V}\mathcal{E} \otimes T^*\mathcal{M}, \tau_{\mathcal{E}} \otimes \tau_{\mathcal{M}}^*, \mathcal{E})$ (cf. [9]). Here, $\mathfrak{V}\mathcal{E}$ denotes the vertical (w.r.t. π) tangent bundle on \mathcal{E} , while $\tau_{\mathcal{E}}$ and $\tau_{\mathcal{M}}^*$ stand for the respective canonical projections. Note that $T^*\mathcal{M}$ is in fact pulled back onto \mathcal{E} —this has been suppressed for simplicity.

Let the dual jet bundle $((\mathfrak{J}^1\mathcal{E})^*, (\pi_0^1)^*, \mathcal{E})$ be the fibre bundle over \mathcal{E} with affine, fibre preserving mappings from $\mathfrak{J}^1\mathcal{E}$ to $\Lambda^n T^*\mathcal{M}$, i.e. for every $v \in \mathcal{E}$ one has

$$((\mathfrak{J}^1\mathcal{E})^*)_v = \{A : (\mathfrak{J}^1\mathcal{E})_v \rightarrow \Lambda^n(T^*\mathcal{M})_{\pi(v)} \text{ affine}\}.$$

$(\mathfrak{J}^1\mathcal{E})^*$ inherits a vector space structure from $\Lambda^n T^*\mathcal{M}$.

As a connection provides a map $\Gamma : \mathcal{E} \rightarrow \mathfrak{J}^1\mathcal{E}$ (see Appendix A for details), there is an isomorphism

$$i_{\Gamma} : \mathfrak{J}^1\mathcal{E} \xrightarrow{\cong} \mathfrak{V}\mathcal{E} \otimes T^*\mathcal{M} \quad (1)$$

that sends the image of Γ to the zero section. This induces a splitting of the dual jet bundle $((\mathfrak{J}^1\mathcal{E})^*)$,

$$(i_{\Gamma})^* : (\mathfrak{J}^1\mathcal{E})^* \xrightarrow{\cong} ((\mathfrak{V}\mathcal{E})^* \otimes T\mathcal{M}) \otimes \Lambda^n T^*\mathcal{M} \oplus \Lambda^n T^*\mathcal{M} =: \mathcal{P} \oplus \mathcal{L}, \quad (2)$$

where we have introduced the notation \mathcal{P} for the first summand and \mathcal{L} for the latter. Observe that \mathcal{L} is a line bundle over \mathcal{E} which is trivial for orientable manifolds. \mathcal{P} will be called the *polysymplectic phase space* from now on. Note that, although the splitting (2) requires the choice of a connection, the projection from $(\mathfrak{J}^1\mathcal{E})^*$ onto \mathcal{P} is canonical.

2.1. Coordinates

Let $(x^{\mu}, v^A)^1$ be local coordinates of \mathcal{E} . Then to every point $j_m^1\varphi$ in $\mathfrak{J}^1\mathcal{E}$ one can assign the matrix elements of $T_m\varphi$ in those coordinates, hence we have locally a set

¹For the rest of this article, lower case Greek indices $\mu, \nu, \rho, \sigma, \dots$ label directions on the base manifold \mathcal{M} , i.e. run from 1 to n , while capital Latin characters A, B, C, D, \dots are used for fibre coordinates of \mathcal{E} , i.e. run from 1 to N .

(x^μ, v^A, v_μ^A) for each such point. Further, the choice of coordinates in \mathcal{E} induces a trivialisation of the dual jet bundle: Every affine map \underline{p} can be written as $\underline{p} : (x^\mu, v^A, v_\mu^A) \mapsto (x^\mu, v^A, (p_A^\mu v_\mu^A + p)d^n x)$, hence (x^μ, v^A, p_A^μ, p) provides a set of local coordinates. Then, using the isomorphism $(i_\Gamma)^*$, (2), one can show [8] that the first three sets of coordinates (x^μ, v^A, p_A^μ) can be used to label \mathcal{P} locally. Furthermore, the arbitrariness of the choice of a connection Γ is projected out, hence the projection $\pi_{\mathcal{P}}$ from $(\mathfrak{J}^1 \mathcal{E})^*$ onto \mathcal{P} is canonical.

2.2. $\bar{\Gamma}$ -vertical differentials

DEFINITION 1. Let $P_{\bar{\Gamma}}$ be the projection of $T\mathcal{P}$ onto its vertical subbundle \mathfrak{VP} constructed from a connection $\bar{\Gamma} : \mathcal{P} \rightarrow \mathfrak{J}^1 \mathcal{P}$, cf. (5.). The corresponding $\bar{\Gamma}$ -vertical differential $d_{\bar{\Gamma}}$ is the map

$$d_{\bar{\Gamma}} : \Omega^r(\mathcal{P}) \rightarrow \Omega^{r+1}(\mathcal{P}), \quad \forall r,$$

satisfying the following properties (d denotes the exterior derivative on \mathcal{P}):

1. For $f \in \mathcal{C}^\infty(\mathcal{P})$, $X \in \mathfrak{X}(\mathcal{P})$, one has $(d_{\bar{\Gamma}} f)(X) = (df)(P_{\bar{\Gamma}} X)$.
2. $d_{\bar{\Gamma}}$ acts as a graded derivation of degree 1. For $\alpha, \beta \in \Omega^*(\mathcal{P})$,

$$d_{\bar{\Gamma}}(\alpha \wedge \beta) = (d_{\bar{\Gamma}} \alpha) \wedge \beta + (-)^{|\alpha|} \alpha \wedge (d_{\bar{\Gamma}} \beta).$$

3. $(d_{\bar{\Gamma}})^2 = 0$.
4. $d_{\bar{\Gamma}}$ vanishes on pulled back forms, i.e. for $\alpha \in \Omega(\mathcal{M})$, $d_{\bar{\Gamma}}((\pi_1)^*)^* \alpha = 0$.

Indeed, one can show that for every $\bar{\Gamma}$ there is a unique $d_{\bar{\Gamma}}$. Note that d and $d_{\bar{\Gamma}}$ do not anticommute.

In what follows, we fix the connection to be the map $\bar{\Gamma}$ defined in the appendix, (16). Then $d_{\bar{\Gamma}}$ is locally given by

$$d_{\bar{\Gamma}} f = \partial_A f \epsilon^A + \partial_\mu^A f \epsilon_A^\mu, \quad d_{\bar{\Gamma}} dx^\mu = 0 = d_{\bar{\Gamma}} \epsilon^A = d_{\bar{\Gamma}} \epsilon_A^\mu, \tag{3}$$

where $f \in \mathcal{C}^\infty(\mathcal{P})$ is a function and the forms ϵ^A and ϵ_A^μ are defined by (18) and together with the dx^μ form a basis of $T^* \mathcal{P}$.

2.3. Canonical forms

As $\mathcal{P} = ((\mathfrak{WE})^* \otimes T\mathcal{M}) \otimes \Lambda^n T^* \mathcal{M}$ and fibres of $\Lambda^n T^* \mathcal{M}$ are one-dimensional, every point of \mathcal{P} can be written as $\alpha \otimes \omega$, where α and ω are elements of $(\mathfrak{WE})^* \otimes T\mathcal{M}$ and $\Lambda^n T^* \mathcal{M}$ respectively. α can be viewed as a map $\mathfrak{WE} \rightarrow T\mathcal{M}$. Define an n -form Θ_Γ on \mathcal{P} by

$$\begin{aligned} & (\Theta_\Gamma)_{(\alpha \otimes \omega)}(X_1, \dots, X_n) \\ &= \sum_\sigma \frac{(-)^\sigma}{n!} \omega(\alpha(P_\Gamma \circ T(\pi_0^1)^*(X_{\sigma(1)})), T(\pi_1)^* X_{\sigma(2)}, \dots, T(\pi_1)^* X_{\sigma(n)}). \end{aligned} \tag{4}$$

As ω is linear in each argument, the right hand side is well defined, i.e. does not depend on the splitting $\alpha \otimes \omega$. Θ_Γ generalises the construction of the canonical 1-form on a cotangent bundle, but due to the fact that \mathcal{E} is non trivial one has to choose a connection.

The *polysymplectic* $(n + 1)$ -form is defined through the action of $d_{\bar{F}}$ on Θ_{Γ} ,

$$\Omega_{\Gamma} = -d_{\bar{F}}\Theta_{\Gamma}. \tag{5}$$

Using coordinates as above, one finds

$$\begin{aligned} \Theta_{\Gamma(x,v,p)} &= \frac{1}{(n-1)!} \epsilon_{\mu\nu_1 \dots \nu_{n-1}} p_A^\mu dv^A \wedge dx^{\nu_1} \wedge \dots \wedge dx^{\nu_{n-1}} + p_A^\mu \Gamma_\mu^A d^n x, \\ \Omega_{\Gamma(x,v,p)} &= \frac{1}{(n-1)!} \epsilon_{\mu\nu_1 \dots \nu_{n-1}} dv^A \wedge dp_A^\mu \wedge dx^{\nu_1} \wedge \dots \wedge dx^{\nu_{n-1}} - d_{\bar{F}}(p_A^\mu \Gamma_\mu^A d^n x). \end{aligned} \tag{6}$$

Note that we have used (17) and that Ω_{Γ} indeed does depend on Γ rather than on Γ . Moreover, the $d_{\bar{F}}$ -cohomology class of Ω_{Γ} is independent of Γ .

3. Covariant Hamiltonian dynamics

DEFINITION 2. Let $d_{\bar{F}}$ be the vertical differential to $\bar{\Gamma}$ of (16). A horizontal form $H \in \Omega(\mathcal{P})$ is called a *Hamiltonian form* if there exists a multi-vector X_H , i.e. an element in $A\mathfrak{X}(\mathcal{P})$, that satisfies

$$d_{\bar{F}}H = X_H \lrcorner \Omega_{\Gamma}. \tag{7}$$

In this case, X is called *Hamiltonian multi-vector field*. The set of all Hamiltonian forms is denoted by \mathcal{H} .

Note that not every horizontal form is Hamiltonian. Rather, condition (7) imposes a severe restriction on such forms.

PROPOSITION 1. [Multimomentum dependence of Hamiltonian forms, [4]]

Let $H \in \mathcal{H}$ be a Hamiltonian form of degree $|H|$. If $|H| > 0$, then, using coordinates as above,

$$H_{(x,v,p)} = \sum_{r=0}^{n-|H|} \frac{1}{(|H|)!} p_{A_1}^{\mu_1} \dots p_{A_r}^{\mu_r} h_{(x,v)}^{A_1 \dots A_r \mu_{r+1} \dots \mu_{n-|H|}} \epsilon_{\mu_1 \dots \mu_n} dx^{\mu_{n+1-|H|}} \dots dx^{\mu_n}, \tag{8}$$

where the functions $h^{A_1 \dots A_r \mu_{r+1} \dots \mu_{n-|H|}}$ are pulled back from \mathcal{E} .

For a proof, we refer to [4, 8]. There is no restriction on functions on \mathcal{P} . Section 5. provides (standard) examples for Hamiltonian $(n - 1)$ -forms.

There is an additional operation on $(\pi_1)^*$ -horizontal forms, namely the lifted Hodge- $*$ operation on \mathcal{M} . Of course, $*$ is invertible.

DEFINITION 3 AND THEOREM . [Gerstenhaber structure [4, 5]] *The maps*

$$\begin{aligned} \{, \} : \mathcal{H} \times \mathcal{H} \ni (F, G) &\mapsto \{F, G\} = (-)^{n-|F|} X_F \lrcorner X_G \lrcorner \Omega_{\Gamma} \in \mathcal{H}, \\ \bullet : \mathcal{H} \times \mathcal{H} \ni (F, G) &\mapsto F \bullet G = *^{-1} (*F \wedge *G) \in \mathcal{H} \end{aligned} \tag{9}$$

are well defined (for the definition of $*$ see the remark above) and satisfy (let $F, G, H \in \mathcal{H}$ be horizontal forms of (exterior) degree $|F|$, $|G|$, and $|H|$, respectively)

$$1. \{F, G\} = -(-)^{(n+1-|F|)(n+1-|G|)} \{G, F\},$$

2. $F \bullet G = (-)^{(n-|F|)(n-|G|)} G \bullet F,$
3. $(-)^{(n+1-|F|)(n+1-|H|)} \{F, \{G, H\}\} + (-)^{(n+1-|G|)(n+1-|F|)} \{G, \{H, F\}\} + (-)^{(n+1-|H|)(n+1-|G|)} \{H, \{F, G\}\} = 0,$
4. $\{F, G \bullet H\} = \{F, G\} \bullet H + (-)^{(n+1-|F|)(n-|G|)} G \bullet \{F, H\}.$

Further, the bracket does not depend on the specific choice of \bar{F} .

A detailed proof can be found in [8]. Note that, in particular, $\{F, G\}$ does not depend on the ambiguity in the correspondence of Hamiltonians F and G and their Hamiltonian multi-vector fields X_F, X_G .

4. Legendre transformation and canonical 2-form on the space of solutions

In this section, we try to establish a link between the multisymplectic formalism and the standard Hamiltonian formulation of field theory. The Euler–Lagrange equations of a given field theory are partial differential equations. Typically a well posed initial value problem for such a set of field equations consists of the specification of two sets of functions

$$(\varphi^A(x), \pi_A(x))_{A=1, \dots, N} \tag{10}$$

on some hypersurface Σ of \mathcal{M} . π_A is known as the momentum variable to the field φ_A . The corresponding solution is a set of functions $\{\Phi^A(x, t)\}$ (t labels the additional coordinate in \mathcal{M}) on \mathcal{M} . The initial data (10) span the field-theoretical phase space \mathcal{P}_{FT} , while the fields Φ span the covariant section space \mathcal{P}_{cov} (the name has been chosen to stress the analogy to the path space of classical mechanics, cf. [1]) which typically is some subspace of $\Gamma(\mathcal{E})$. Despite the similarity in the names, \mathcal{P}_{FT} should not be confused with the polysymplectic phase space, \mathcal{P} . In particular, both the field theoretical phase space and the section space are infinite-dimensional spaces, while \mathcal{P} is of dimension $(n + 1)(N + 1) - 1$. The Legendre map provides a map from the section space to \mathcal{P}_{FT} , which is explicitly given by

$$\varphi^A(x) = \Phi^A(x, 0), \quad \pi_A(x) = \frac{\partial L}{\partial \partial_t \Phi^A} ((x, 0), \Phi(x, 0), \Phi'(x, 0)), \tag{11}$$

where L is the Lagrange density and Φ' denotes all derivatives of the fields Φ . On the field theoretical phase space there is a canonical (weak symplectic) 2-form that can be pulled back using the tangent map of (11). Thus there is a (noncanonical) 2-form on ω_L on the section space \mathcal{P}_{cov} . The vector fields which can be plugged into ω_L are infinitesimal (vertical) variations of the fields Φ . They form the tangent space of \mathcal{P}_{cov} (we leave problems arising from the infinite dimensionality aside), which therefore consists of all sections of the vertical bundle \mathfrak{VE} to \mathcal{E} . The explicit expression for the contraction of ω_L with two such vectors X, Y is found to be (cf. [10] for details)

$$\omega_L(X, Y) = \frac{1}{2} \int_{\Sigma} \left(\frac{\partial^2 L}{\partial \varphi^A \partial \varphi^B} (X^A Y^B - Y^A X^B) + \frac{\partial^2 L}{\partial \varphi^A \partial \varphi^B} (Y^B \nabla_{\mu} X^A - X^B \nabla_{\mu} Y^A) \right) d_{\nu} x, \tag{12}$$

where ∇ denotes a covariant derivative on sections of \mathcal{E} (and hence on those of \mathfrak{VE}). The ν -summation of this formula reduces to the t -component if the normal direction of Σ

happens to be the time coordinate. On the other hand, let \tilde{X} be the prolongation to the vertical bundle \mathfrak{VE} of a given variational field X . Explicitely, one finds

$$\tilde{X}^A(x, t) = (X^A, \nabla_\mu X^A(x, t)), \tag{13}$$

where we have used the Christoffel symbols of ∇ for the map i_Γ , (1). Now one can use the Legendre transformation and the connection Γ to pull back the polysymplectic $(n + 1)$ -form onto \mathfrak{VE} . The contraction of the resulting $(n + 1)$ -form with two prolonged variational fields \tilde{X}, \tilde{Y} yields a horizontal $(n - 1)$ -form, which can be integrated over Σ . This yields back the expression (12). Indeed, a calculation shows

$$\begin{aligned} ((\mathfrak{FL})^* \Omega_\Gamma)_{(x,v,v')} &= (\partial_B \partial_A^\mu L) dv^A \wedge dv^B \wedge d_\mu x + (\partial_B^\nu \partial_A^\mu L) dv^A \wedge dv_\nu^B \wedge d_\mu x \\ &\quad - (\Gamma_\mu^A (\partial_B \partial_A^\mu L) dv^B + \Gamma_\mu^A (\partial_B^\nu \partial_A^\mu L) dv_\nu^B + (\partial_A^\mu L) \partial_B \Gamma_\mu^A - (\partial_\mu \partial_A^\mu L) dv^A) \wedge dx, \end{aligned}$$

where (\mathfrak{FL}) denotes the Legendre map and the repeated use of the map i_Γ in the pull back has been suppressed in the notation. Note that the last line vanishes on two vertical vectors which establishes the following result.

LEMMA 1. *Let X, Y be field variations on \mathcal{M} , i.e. π -vertical tangent vector fields on \mathcal{E} . Let \tilde{X}, \tilde{Y} be their lifts via the Legendre mapping (11) – taken for arbitrary value of t – to \mathcal{P} . Then on every section $\Phi \in \Gamma(\mathcal{E})$ we find*

$$\omega_L(X \circ \Phi, Y \circ \Phi) = \int_\Sigma \tilde{\Phi}^* (\tilde{Y} \lrcorner (\tilde{X} \lrcorner \Omega_\Gamma)), \tag{14}$$

where $\tilde{\Phi}$ denotes the prolongation of Φ to \mathcal{P} using (11) again.

The lemma shows how the standard Hamiltonian formulation of field theory and the polysymplectic approach can be related. Objects (vector fields in this case) in polysymplectic fields theory have to be evaluated on (prolonged) sections and integrated over the submanifold Σ that has been chosen for the Hamiltonian picture. For instance, the variations X and Y are given for all sections, but when evaluated on a specific Φ , they become variations of that Φ .

5. Internal symmetries

From the peculiar momentum dependence of multisymplectic Hamiltonian $(n - 1)$ -forms as stated in Proposition 1 one immediately concludes the following.

LEMMA 2. *Let X be a Hamiltonian 1-vector field in the sense of (7). Then X is projectable along $(\pi_0^1)^*$ onto \mathcal{E} . Conversely, the canonical lift \hat{X} to \mathcal{P} of a π -vertical vector field X on \mathcal{E} is Hamiltonian. Its Hamiltonian form is given by $I(X) = \hat{X} \lrcorner \Theta_\Gamma$.*

Proof: According to Proposition 1, a general Hamiltonian $(n - 1)$ -form H can be written as

$$H(x, v, p) = \frac{1}{(n - 1)!} (p_A^\mu f^A(x, v) + g^\mu(x, v)) \epsilon_{\mu\nu_1 \dots \nu_{n-1}} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_{n-1}},$$

where f^A, g^μ are pulled back functions from \mathcal{E} . Thus the corresponding Hamiltonian vector field X takes the form

$$X_{(x,v,p)} = f^A(x, v)\partial_A - (p_A^\mu \partial_B f^A(x, v) + \partial_B g^\mu)\partial_\mu^B,$$

so X is projectable as stated.

Let X be a vertical vector field on \mathcal{E} . X generates a one-parameter group of fibre diffeomorphisms on \mathcal{E} which by jet prolongation can be transported to $\mathfrak{J}^1\mathcal{E}$ (see [3], Section 4B). As $(\mathfrak{J}^1\mathcal{E})^*$ is affine dual to the first jet bundle, one can lift this action further by pull-back and—using the canonical projection $\pi_{\mathcal{P}}$ —transport it to \mathcal{P} . For a given vector field $X = X^A\partial_A$ the coordinate expression of the latter is as follows

$$\hat{X}_{(x,v,p)} = X_{(x,v)}^A\partial_A - p_B^\nu \partial_A X_{(x,v)}^B \partial_\nu^A. \tag{15}$$

One immediately verifies that such an \hat{X} is Hamiltonian with Hamiltonian $(n - 1)$ -form $I(X)$:

$$d_{\bar{F}}I(X) = \frac{1}{(n - 1)!} \epsilon_{\mu\nu_1 \dots \nu_{n-1}} (\partial_B X^A e^B + X^A e_A^\mu) dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{n-1}} = \hat{X} \lrcorner \Omega_\Gamma. \quad \square$$

Further, as the lifting of two such vector fields X, Y is a Lie-homomorphism, $[\widehat{X}, \widehat{Y}] = [\hat{X}, \hat{Y}]$, and $\hat{X} \lrcorner \hat{Y} \lrcorner \Omega_\Gamma = [\hat{X}, \hat{Y}] \lrcorner \Theta_\Gamma$, one has

$$\{I(X), I(Y)\} = \hat{X} \lrcorner \hat{Y} \lrcorner \Omega_\Gamma = [\hat{X}, \hat{Y}] \lrcorner \Theta_\Gamma = I([X, Y]).$$

Hence the map I provides a momentum map for vertical symmetries of \mathcal{E} .

One might wonder what the different ways to lift vertical automorphisms to the jet bundle and to its affine dual have to do with each other (cf. (15) to (13)). The link is provided by the Legendre transformation: If an automorphism leaves the Lagrange density invariant, then its lift to the first jet bundle is mapped to its lift to the co-jet bundle via the Legendre transformation.

A. Some facts about connections [7]

Most generally, a connection on a fibre bundle $(\mathcal{E}, \pi, \mathcal{M})$ is a projection onto the vertical subbundle $\mathfrak{V}\mathcal{E}$. One can show that such projections are in 1 – 1 correspondence with maps $\Gamma : \mathcal{E} \rightarrow \mathfrak{J}^1\mathcal{E}$. In coordinates, one writes

$$\Gamma : \mathcal{E} \ni (x^i, v^A) \mapsto (x^i, v^A, \Gamma_i^A(x, v)) \in \mathfrak{J}^1\mathcal{E}.$$

For $(\mathcal{E}, \pi, \mathcal{M})$ being a vector bundle and Γ a linear connection (cf. [7]), one has in addition $\Gamma_i^A(x, v) = -\Gamma_{iB}^A(x)v^B$, where $\Gamma_{iB}^A(x)$ denote the Christoffel symbols of the corresponding covariant derivative.

B. A connection $\bar{\Gamma}$ on \mathcal{P}

If we want to construct a connection on \mathcal{P} , then we are looking for a $(\pi_0^1)^*$ -fibre preserving map $\bar{\Gamma} : \mathcal{P} \rightarrow \mathfrak{J}^1\mathcal{P}$, where $\mathfrak{J}^1\mathcal{P}$ is defined w.r.t. the projection $(\pi_1)^*$ onto \mathcal{M} . For such a map, one needs both a connection Γ on \mathcal{E} and a connection Λ on $T\mathcal{M}$ (cf. [6]). The coordinate expression of the desired map $\bar{\Gamma}$ turns out to be (cf. [8])

$$\bar{\Gamma} : (x^\mu, v^A, p_A^\mu) \mapsto (x^\mu, v^A, p_A^\mu, \Gamma_\mu^A, (\partial_A \Gamma_\mu^B)\delta_\sigma^\nu p_B^\sigma + \Lambda_{\mu\sigma}^\nu \delta_A^B p_B^\sigma - \Lambda_{\mu\rho}^\rho \delta_A^B \delta_\sigma^\nu p_B^\sigma). \tag{16}$$

Since $T\mathcal{M}$ is a vector bundle we can use the Christoffel symbols $A_{\mu\nu}^\rho$ of A . Moreover,

$$(\partial_A \Gamma_\mu^B) \delta_\sigma^\mu + A_{\mu\sigma}^\mu \delta_A^B - A_{\mu\sigma}^\rho \delta_A^B \delta_\sigma^\mu = (\partial_A \Gamma_\sigma^B) + (A_{\mu\sigma}^\mu - A_{\sigma\mu}^\mu) \delta_A^B, \quad (17)$$

but the last term vanishes for torsion free connections A . With the help of this connection, we can define the a basis on $T^*\mathcal{P}$ at every point (x, v, p) by

$$dx^\mu, \quad e^A = dv^A - \Gamma_\mu^A(x, v) dx^\mu, \quad e_A^\mu = dp_A^\mu - \bar{\Gamma}_{\nu A}^\mu(x, v, p) dx^\nu. \quad (18)$$

REFERENCES

- [1] R. Abraham and J. E. Marsden: *Foundations of Mechanics*, Addison-Wesley, New York 1978.
- [2] J. Dito: *Lett. Math. Phys.* **20** (1990), 125–134.
M. Dütsch and K. Fredenhagen: *Algebraic Quantum Field Theory, Perturbation Theory, and the Loop Expansion*, hep-th/0001129.
H. Römer and C. Paufler: Anomalies and Star Products, in *Quantum Theory and Symmetries*, H.-D. Doebner, J.-D. Hennig, W. Lücke, and V. K. Dobrev eds., World Scientific, Singapore 2000.
- [3] M. J. Gotay, J. Isenberg and J. E. Marsden: *Momentum Maps and Classical Relativistic Fields I: Covariant Field Theory*, physics/9801019.
- [4] I. V. Kanatchikov: *Rep. Math. Phys.* **40** (1997), 225–234. hep-th/9710069.
- [5] I. V. Kanatchikov: *Rep. Math. Phys.* **41** (1998), 49–90, hep-th/9709229.
- [6] I. Kolář: *J. Nat. Acad. Math. India* **5** (1987), 127–141.
- [7] I. Kolář, P. W. Michor and J. Slovák: *Natural Operations in Differential Geometry*, Springer, Berlin, Heidelberg, New York 1993.
- [8] C. Paufler: *Rep. Math. Phys.* **47** (2001), 101–119.
- [9] D. J. Saunders: *The Geometry of Jet Bundles*, Lond. Math. Soc. Lect. Note Ser. **142**, Cambridge Univ. Press, Cambridge 1989.
- [10] N. M. J. Woodhouse: *Geometric Quantization*, Clarendon Press, Oxford 1992.