Theoretical Physics 3 — Quantum Mechanics

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Problem Sheet No. 4

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Problem 7. The Potential Well (20 Points) We consider a one-dimensional square potential well with finite depth, described by the potential

$$V(x) = \begin{cases} 0 & x < -\frac{L}{2}\,, \\ -V_0 & -\frac{L}{2} \le x \le \frac{L}{2}\,, \\ 0 & x > \frac{L}{2}\,. \end{cases} \tag{1}$$

(a) Find the equations that determine the energy levels of the symmetric and anti-symmetric bound states, respectively. In order to do this, solve the time-independent Schrödinger equation for the case $-V_0 < E < 0$ in the three regions of constant potential and use conditions for the continuity and differentiability at the boundaries $|x| = L/2$. **3P**

(b) Use the results of (a) to clarify if there is a minimal depth V_{min} of the potential well required for the existence of a bound state. How many bound states are there in general for a given depth V_0 and mass m of the particle in the potential well? Sketch the characteristics of the wave function (e.g., its modulus squared) for a typical solution. **3P**

(c) Consider half of the above potential well, i.e. $V(x > 0)$ as above, but $V(x < 0) = \infty$. Is there, independent of V_0, always a bound state in this case? **1P**

In the following we want to study the scattering of a particle at a potential well. A solution of the Schrödinger equation for positive energy $E > 0$ describing this situation can be written in the following form:

$$\Psi(x) = \begin{cases} e^{ik_0x} + \alpha_- e^{-ik_0x} & x < -\frac{L}{2}\,, \\ \beta_+ e^{ikx} + \beta_- e^{-ikx} & -\frac{L}{2} \le x \le \frac{L}{2}\,, \\ S e^{ik_0x} & x > \frac{L}{2}\,, \end{cases} \tag{2}$$

where $k_0 = \sqrt{2mE}/\hbar$ and $k = \sqrt{2m(E+V_0)}/\hbar$. This solution describes a plane wave approaching the potential well from the left, while part of it is reflected (amplitude α_-) and part of it is transmitted across the well to the right (amplitude $S = S(E)$). The quantity $S(E)$ is called scattering amplitude and its modulus squared $T(E) = |S(E)|^2$ is the probability for transmission. The modulus squared of α_- is the probability for reflection, $R(E) = |\alpha_-|^2$.

(d) Using the conditions for the continuity of the wave function and its derivative at $|x| = L/2$, determine the coefficients α_-, β_+ and β_- as well as the scattering amplitude $S(E)$. **3P**

(e) Show that the probability for transmission can be written in the form

$$T(E) = \left(1 + \frac{1}{4}\frac{V^2}{E(E+V)}\sin^2(kL)\right)^{-1} \tag{3}$$

and draw a sketch of its characteristic behaviour as a function of E. Show that the interpretation of the wave function as a probability amplitude is indeed consistent by proving that

$$R(E) + T(E) = 1\,. \tag{4}$$

3P

(f) For which energies E_n does one observe resonances ($T(E_n) = 1$) and what is the amplitude of the reflected wave in this case? Calculate the wave length $\lambda = 2\pi/k$ inside the potential well for the resonance energies and compare your result with the case of bound states in the infinitely deep potential well. **2P**

(g) For positive energies, $S(E)$ is finite; however, for negative energies there are poles. Show that poles of $S(E)$ appear at energies which obey one of the following two equations:

$$k\cot\left(\frac{kL}{2}\right) = -k_0\,, \qquad k\tan\left(\frac{kL}{2}\right) = k_0\,, \tag{5}$$

where $k_0 = \sqrt{-2mE}/\hbar$. Compare these conditions with those for the energy levels of the bound states of problem part (a). **2P**

(h) Expand $S(E)^{-1}$ in the vicinity of a resonance up to first order in $(E - E_n)$ and determine, in this approximation, the position of the pole of $S(E)$ of the corresponding pole in the complex E-plane. Show that the probability for transmission in this approximation follows the profile of a Cauchy-Lorentz distribution (non-relativistic Breit-Wigner distribution)

$$T(E) \sim \frac{1}{(E - E_n)^2 + \frac{\Gamma^2}{4}} \tag{6}$$

and determine Γ. The results of this investigation show that resonances can be considered as almost-bound states with a lifetime approximated by the inverse of the decay width $\hbar/\Gamma$. **3P**

Please tell us how much time you needed to solve the problems.