Canonical Group Quantization

Why Study Quantization?

- Conventional Quantization works only for $\mathbf{R}^n$. A more general quantization method is needed to handle symmetries and non-trivial phase spaces (e. g. in quantum gravity).
- We want to understand the link between quantum systems and their classical counterparts.
- How do topological properties of the phase space affect quantum theory? (This is especially interesting with respect to the *spin-statistics theorem*.)
- What are the characteristics of quantum theory?

Conventional Quantization

- Describe pure states by rays in Hilbert space $\mathcal{H} = L^2(\mathbf{R}^n, dx)$.
- Replace q_j and p_j by self-adjoint operators $\hat{q}_i$ and $\hat{p}_j$ such that the canonical commutation relations (CCR) hold:

$$[\hat{q}_i, \hat{q}_j] = 0 = [\hat{p}_i, \hat{p}_j] \,, \quad [\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}\mathbf{1} \,.$$

- Quantize other observables $f(q_i, p_j)$ by $f(\hat{q}_i, \hat{p}_j)$. Operator ordering poses some problems here.

Dirac's Quantization Map

$$\underset{C^\infty(M,\mathbf{R})}{\text{Classical Observables}} \xrightarrow[\mathcal{Q} = \hat{}]{} \underset{\mathrm{Op}(\mathcal{H})}{\text{Quantum Operators}}$$

Q1 $\mathcal{Q}$ is $\mathbf{R}$-linear,

Q2 $\mathcal{Q}(f)$ is essentially self-adjoint,

Q3 $\mathcal{Q}$ maps Poisson brackets to commutators:

$$[\mathcal{Q}(f), \mathcal{Q}(g)] = i\hbar\mathcal{Q}(\{f, g\}_{\mathrm{PB}}) \,,$$

Q4 $\mathcal{Q}$ is irreducible:
If $\{f_1, \ldots, f_k\}$ is a complete set of observables, then $\{\mathcal{Q}(f_1), \ldots, \mathcal{Q}(f_k)\}$ is a complete set of operators,
(i. e., if A commutes with all $\mathcal{Q}(f_k)$, then $A \propto \mathbf{1}$)

Q5 $\mathcal{Q}(1_M) = \mathbf{1}_{\mathcal{H}}$ (excludes some trivial solutions).

Problem! Groenewold, van Hove, and others showed that $\mathcal{Q}$ doesn't exist!

Possible Ways Out …

The best we can achieve is a compromise:

- In **Deformation Quantization** one modifies the correspondence between commutators and Poisson brackets by adding terms in higher orders of $\hbar$.

  The Lie algebra structure is linked directly to the *geometry* of the system. We want to keep this.
- In **Geometric Quantization** one restricts the set of quantizable observables.

  This is physically acceptable, since even on a classical level *we can only measure a small set of observables.*
- ???

But: How do we construct the set of quantizable observables?

The Canonical Group

The CCR are the commutators of a Lie algebra. For the classical configuration space $Q = \mathbf{R}$ it is $\mathscr{L}H_1 = \mathbf{R}^2 \times \mathbf{R}$ (known as *Weyl-Heisenberg algebra*) with Lie bracket:

$$[(a_1, b_1; c_1), (a_2, b_2; c_2)] := (0, 0; b_1a_2 - b_2a_1) \,.$$

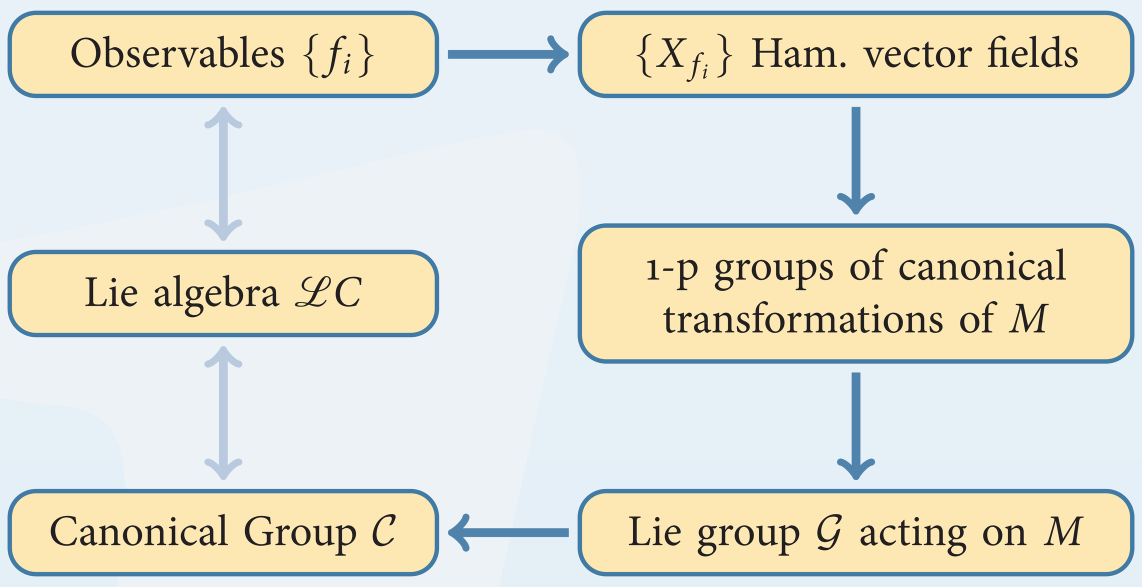
Using $(a, b; c) \mapsto \frac{1}{i}(a\hat{p} + b\hat{q} + c\hbar\mathbf{1})$ this is equivalent to the usual commutation relations above.

The CCR can be encoded in a Lie group, the so-called Canonical Group $\mathcal{C}$.

In conventional quantum mechanics, $\mathcal{C}$ is the Heisenberg group.

Origin of the Canonical Group

Crucial idea: Any complete set of observables generates a group $\mathcal{G}$ which acts by canonical transformations on phase space!



Problem: In conventional quantum mechanics the functions q and p form a complete set of observables. But what are their analogues in case of a more general phase space M?

Solution: Reverse the procedure, i. e. find a group $\mathcal{G}$ that acts by canonical transformations on M and generates enough observables. Considered from this point of view the commutation relations are deeply rooted in the geometry of phase space!

Isham's Quantization Scheme (Part 1)

We start with a group $\mathcal{G}$ that acts on phase space M in a 'suitable way'. Among other things we require that the *fundamental* vector field $\gamma(A)$ of any element $A \in \mathscr{L}\mathcal{G}$ is at the same time a *Hamiltonian* vector field $j(f)$ for some function $f \in C^\infty(M, \mathbf{R})$. This can be summarized in the following commutative diagram:

$$0 \longrightarrow \mathbf{R} \hookrightarrow C^\infty(M, \mathbf{R}) \xrightarrow{j} \mathfrak{X}_{\mathrm{Ham}}(M) \longrightarrow 0 \,, \qquad P: \mathscr{L}\mathcal{G} \to C^\infty(M,\mathbf{R}) \,, \quad \gamma: \mathscr{L}\mathcal{G} \to \mathfrak{X}_{\mathrm{Ham}}(M)$$

where the maps are given by:

$$j(f) = -X_f = -df^\sharp \,, \quad \gamma(A)_x = \frac{d}{dt}\Big|_{t=0} \Phi_{\exp(-tA)}x \,.$$

The so-called *Poisson map* P has to fulfil:

$$\gamma(A) = j \circ P(A) = -X_{P(A)} \,,$$

which is essentially the requirement from above.

If P is a Lie algebra homomorphism we can continue to the second part of the procedure, but sometimes we run into problems. This happens when the obstruction cocycle:

$$z(A, B) := \{P(A), P(B)\}_{\mathrm{PB}} - P([A, B])$$

cannot be made to vanish, i. e. if it is not a coboundary. In this case we have to consider the *central extension* $\mathscr{L}\mathcal{C}$ of $\mathscr{L}\mathcal{G}$ as in the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{R} & \hookrightarrow & C^\infty(M, \mathbf{R}) & \xrightarrow{j} & \mathfrak{X}_{\mathrm{Ham}}(M) & \longrightarrow & 0 \\ & & & & \uparrow P' & \nwarrow P & \uparrow \gamma & & \\ 0 & \longrightarrow & \mathbf{R} & \hookrightarrow & \mathcal{E} = \mathscr{L}\mathcal{C} & \longrightarrow & \mathscr{L}\mathcal{G} & \longrightarrow & 0 \end{array}$$

Afterwards P' is a Lie algebra homomorphism and thus properly maps commutators to Poisson brackets.

Isham's Quantization Scheme (Part 2)

Now that we have the canonical group $\mathcal{C}$ we can study its unitary irreducible representations to obtain quantum operators together with the space of wave functions (although we also consider representations on cross sections of vector bundles).

In general there may be several inequivalent possibilities with different physical meaning, but in case of the Heisenberg group the *Stone-von Neumann theorem* states that all inequivalent representations can be parametrised by a real number $\mu \in \mathbf{R}$.

Since μ's physical unit is that of an action, it is natural to set $\mu = \hbar$, which fixes a scale that discriminates between macroscopic and microscopic effects.

Example: Conventional Quantum Mechanics

If we consider 1-dimensional quantum mechanics, the configuration space is $Q = \mathbf{R}$, whereas the phase space is $M = T^*Q \cong \mathbf{R}^2$. In this case a natural choice is $\mathcal{G} = (\mathbf{R}^2, +)$ which acts on M by translations:

$$\Phi : \mathbf{R}^2 \times M \to M \,, \; \Phi_{(u,v)}(q, p) := (q + u, p - v) \,,$$

as depicted in figure 1.

The corresponding Lie algebra is $\mathscr{L}\mathcal{G} = (\mathbf{R}^2, [\cdot, \cdot])$ with vanishing Lie bracket $[A, B] = 0$ for all $A, B \in \mathscr{L}\mathcal{G}$.

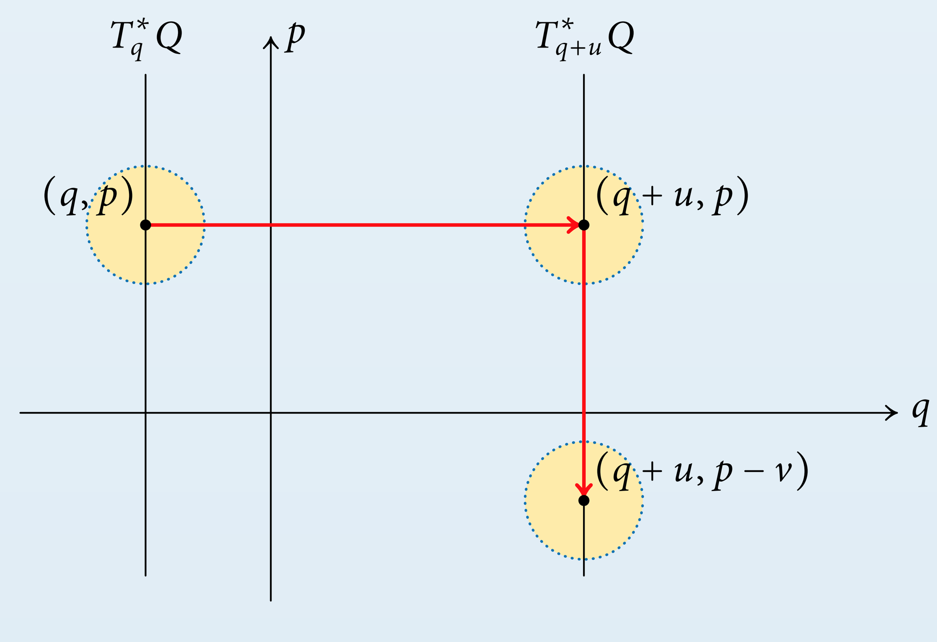


Figure 1: Group $\mathcal{G} = (\mathbf{R}^2, +)$ acting on $M = T^*\mathbf{R}$.

Next we calculate the fundamental vector fields:

$$\gamma(a, b)_{(q,p)} = \frac{d}{dt}\Big|_{t=0} \Phi_{-t(a,b)}(q, p) = -a\partial_q + b\partial_p \,.$$

On the other hand, the Hamiltonian vector field for some function $f \in C^\infty(M, \mathbf{R})$ is given by:

$$j(f) = -X_f = -(\partial_i f)dx^{i\,\sharp} = -(\partial_p f)\partial_q + (\partial_q f)\partial_p \,.$$

Matching both via $j \circ P(A) = \gamma(A)$ yields the Poisson map:

$$P(a, b) := ap + bq + const(a, b) \,.$$

Unfortunately the obstruction cocycle is non-zero:

$$z\big((a_1, b_1), (a_2, b_2)\big) = b_1a_2 - b_2a_1 \,,$$

hence P is not a Lie algebra homomorphism and we have to consider the central extension of $\mathscr{L}\mathcal{G}$, which is $\mathscr{L}\mathcal{C} = \mathbf{R}^2 \times \mathbf{R}$ with Lie bracket:

$$\big[(a_1, b_1; c_1), (a_2, b_2; c_2)\big] = (0, 0, b_1a_2 - b_2a_1) \,.$$

This is the *Weyl-Heisenberg algebra* that describes the CCR of conventional quantum mechanics. The parameter $\hbar$ appears when we study the unitary irreducible representations of $\mathcal{C}$.

Generalizing the Example…

If the phase space M is a cotangent bundle $M = T^*Q$, a good place to start is the group $C^\infty(Q, \mathbf{R})/\mathbf{R} \rtimes \mathrm{Diff}(Q)$ which has almost all the properties we need. It only is much too big. In order to get a useful quantum theory we have to select a subgroup $W \rtimes G$ which can act as the geometric group $\mathcal{G}$ from above. Seen from this point of view, figure 1 is just a special case of the more general figure 2.

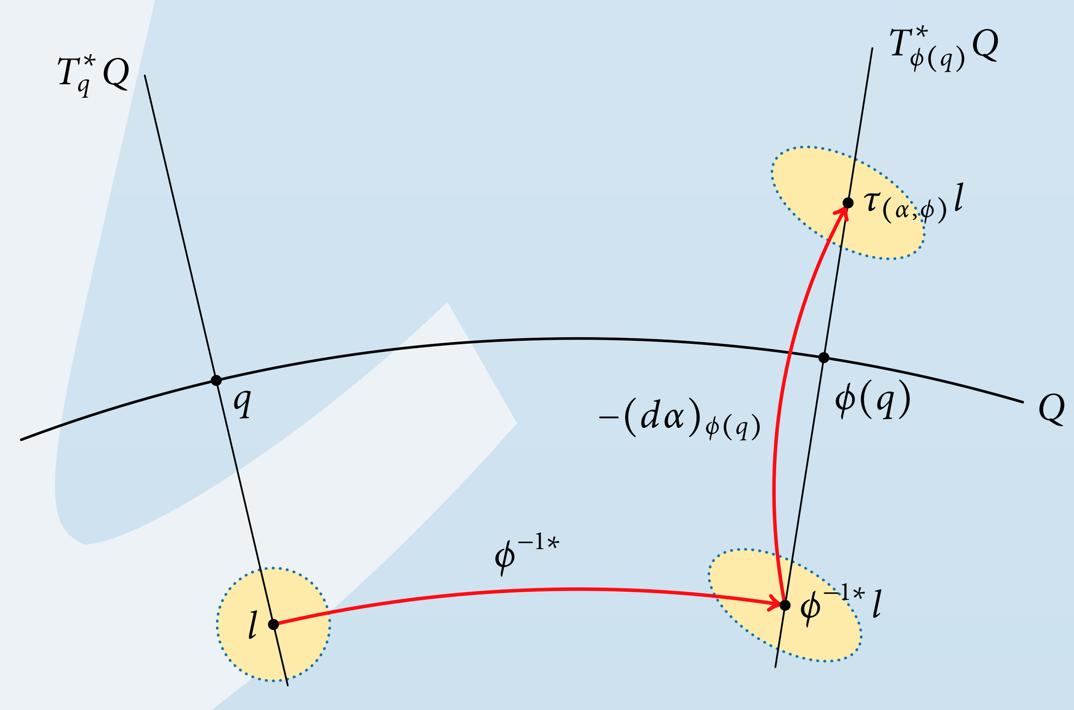


Figure 2: Group action of $C^\infty(Q, \mathbf{R})/\mathbf{R} \rtimes \mathrm{Diff}(Q)$.

Theoretical High Energy Physics
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