Quantum Field Theory I

Wahlpflichtvorlesung
Winter Semester 2009/2010

Fachbereich 08 — Physik, Mathematik und Informatik

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Prologue

This is a script for a first-time course on Quantum Field Theory, given to students who are familiar with concepts of classical and quantum mechanics, special relativity, electromagnetism, and basics of group theory. The course consists of approximately 22 lectures and 6 seminars (practical sessions), each of two academic hours.

This script is supposed to represent the content of the lectures, but is no replacement for your own notes. Equations that need to be derived by the students during the practical sessions are highlighted in blue. In the end of each chapter one can find the literature for further reading, as well as, the homework exercises.

The homework is due at the first lecture that comes after a seminar.

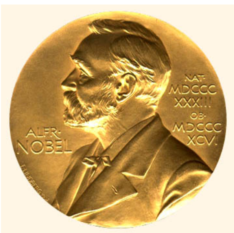
Contents

1

Motivation

Quantum field theory (QFT) is a rigorous and beautiful framework for describing microscopic excitations, elementary particles and waves. It is primarily used to describe the elementary particles and the fundamental interactions between them, vz. the Standard Model, but is also used in condensed matter physics, astrophysics and cosmology, atomic and nuclear physics. It encompasses the laws of quantum mechanics and relativity in a self-consistent way and allows for a systematic study of fundamental symmetries and mechanisms of their breaking. It is the only framework which consistently describes the production and annihilation of matter.

1.1 The Nobel history of QFT



Perhaps the easiest way to demonstrate the importance of QFT in physics is to appreciate how many Nobel Prizes have been given for discoveries made thanks to development of QFT.

QFT began to emerge in the 1920's with attempts to quantize Maxwell's electromagnetic field. The prediction of antiparticles by Dirac's relativistic theory of the electron and the experimental discovery of positrons and the process of annihilation, marked by the Nobel Prizes,

1933 P. A. M. Dirac, "for the discovery of new productive forms of atomic theory."

1936 C. D. Anderson, "for his discovery of the positron."

turned the theory of quantum fields into a mainstream research topic.

The next important milestones were marked by two more Nobel Prizes, 1955 W. C. Lamb, “for his discoveries concerning the fine structure of the hydrogen spectrum,” and P. Kusch, “for his precision determination of the magnetic moment of the electron.”

1959 E. G. Segrè and O. Chamberlain, “for their discovery of antiprotons.”

The work on quantization of Maxwell’s and Dirac’s fields finally led to one of the most successful and elegant theories of our days — the Quantum Electrodynamics (QED), celebrated with a Nobel Prize of

1965 R. P. Feynman, J. Schwinger, and S.-I. Tomonaga “for fundamental work in quantum electrodynamics, with deep-ploughing consequences for the physics of elementary particles.”

The next big success for QFT came with the prediction and subsequent discovery of the massive gauge bosons ($W^{\pm}$, Z^0), which led to a unification of weak and electromagnetic interactions — electroweak gauge theory. Several Nobel Prizes were awarded to honor these achievements:

1979 S.L. Glashow, A. Salam and S. Weinberg, “for their contributions to the theory of the unified weak and electromagnetic interaction between elementary particles, including inter alia the prediction of the weak neutral current.”

1984 C. Rubbia and S. van der Meer, “for their decisive contributions to the large project, which led to the discovery of the field particles W and Z, communicators of weak interaction.”

1999 G. ’t Hooft and M. Veltman, “for elucidating the quantum structure of electroweak interactions in physics.”

The story of electroweak theory would not be complete without these Nobel Prizes awarded for experimental discoveries:

1980 J. W. Cronin and V. L. Fitch, “for the discovery of violations of fundamental symmetry principles in the decay of neutral K-mesons.”

1988 L. M. Lederman, M. Schwartz, and J. Steinberger, “for the neutrino beam method and the demonstration of the doublet structure of the leptons through the discovery of the muon neutrino.”

1995 M. L. Perl, “for the discovery of the tau lepton,” and F. Reines, “for the detection of the neutrino. ”

The invention of Quantum Chromodynamics (QCD) in 1973 and its establishment as the fundamental theory of the strong interactions had been made possible due to work of thousands of physicists. The relevant Nobel Prizes include:

1968 L. W. Alvarez, "for his decisive contributions to elementary particle physics, in particular the discovery of a large number of resonance states, made possible through his development of the technique of using hydrogen bubble chamber and data analysis."

1969 M. Gell-Mann, "for his contributions and discoveries concerning the classification of elementary particles and their interactions."

1976 B. Richter and S. C. Ting, "for their pioneering work in the discovery of a heavy elementary particle of a new kind."

1990 J. I. Friedman, H. W. Kendall, and R. E. Taylor, "for their pioneering investigations concerning deep inelastic scattering of electrons on protons and bound neutrons, which have been of essential importance for the development of the quark model in particle physics."

2004 D. J. Gross, H. D. Politzer and F. Wilczek, "for the discovery of asymptotic freedom in the theory of the strong interaction."

2008 Y. Nambu, "for the discovery of the mechanism of spontaneous broken symmetry in subatomic physics," and M. Kobayashi, T. Maskawa, "for the discovery of the origin of the broken symmetry which predicts the existence of at least three families of quarks in nature. "

It is well-possible this list will grow, as one of the most challenging problems of all time, the problem of color confinement and spontaneous chiral symmetry breaking, is still not solved within QCD.

The electroweak theory together with QCD constitute the Standard Model of particle physics, the best theory we have now for describing the elementary particles and forces among them. However, the role of QFT is by no means limited to the Standard Model. QFT is extensively used in condensed matter physics, cosmology, in practically every branch of modern theoretical physics. As an example I would first of all point to the Nobel Prize of

1982 K. G. Wilson, "for his theory for critical phenomena in connection with phase transitions."

This course will help you to understand some of these discoveries in their full glory.

1.2 Literature

1. Nobel Lectures, `http://nobelprize.org/nobelfoundation/publications/lectures/index.html`.
2. The Nobel Prize Internet Archive, `http://almaz.com/nobel/physics`.
3. For a much deeper historical introduction, see S. Weinberg, "The Quantum Theory of Fields," Volume 1, Chapter 1.

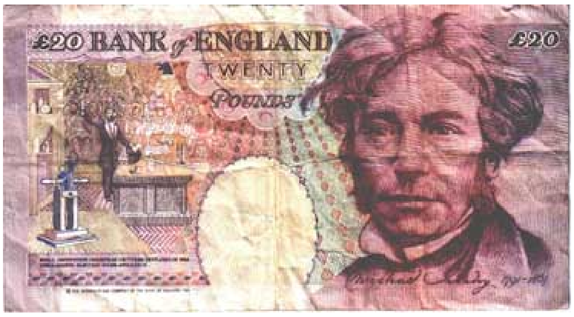
1.3 Exercise

[6 pts] Investigate three of the above-mentioned Nobel Prizes and write **your own** laudation to them (i.e., write a concise formulation of what the prize is awarded for).

2

Classical fields

Originally the concept of a field as physical reality was introduced in the 19th century by Michael Faraday in his description of electricity and magnetism. It did not receive much attention until the works of J. C. Maxwell on laws of electromagnetism, now known as “Maxwell equations”.



In contemporary physics the concept of a field is used not only to describe electromagnetism but basically everything that “lives” in spacetime — particles of matter, forces between them, collective excitations, even the vacuum.

In this chapter we shall introduce the mathematical description of relativistic fields, discuss their properties under the coordinate transformations, and consider the equations which describe their propagation.

2.1 The concept

A *field* is a generic physical entity that “lives” in space $\vec{x}$ and time t, can carry momentum $\vec{p}$ and energy E, and, possibly, has intrinsic degrees of freedom such as spin. Mathematically, a field is a function of space and time coordinates:

$$\varphi = \varphi_k(\vec{x}, t). \tag{2-1}$$

The index k runs over the intrinsic degrees of freedom, e.g., spin polarizations.

Since we are going to deal with relativistic theories, where the time are space are treated equally, we adopt the four-vector notation for the space-time coordinates:

$$x^\mu = (c\,t, \vec{x})\,, \quad \mu = 0, 1, 2, 3. \tag{2-2}$$

and assume that the fields live in flat Minkowski space-time with metric:

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{2-3}$$

a scalar product,

$$x \cdot y = \eta_{\mu\nu} x^\mu y^\nu = x^\mu y_\mu = x_\mu y^\mu = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3\,. \tag{2-4}$$

and an infinitesimal interval:

$$dx^2 = c^2 dt^2 - d\vec{x}^2. \tag{2-5}$$

A field is then simply written as $\varphi_k(x)$, and can be classified according to how it transforms under Lorentz transformations. The most notable examples of relativistic fields are the electromagnetic and gravitational fields, denoted as $A_\mu(x)$ and $g_{\mu\nu}(x)$. They are, respectively, a Lorentz vector and a rank-2 tensor.

2.2 Analogies with classical mechanics

The dynamics, the evolution of field's energy, momentum and intrinsic degrees of freedom, can be described by a Hamiltonian — a functional of the field and its *conjugate momentum* $\pi(x)$:

$$H = H[\varphi(x), \pi(x)]. \tag{2-6}$$

An expression for the Hamiltonian is all we need to specify a particular field theory (FT).

Recall that in classical mechanics (CM) the Hamiltonian is a functional of coordinates and momenta of the system: $H = H[q(t), p(t)]$. Much can be taken over from CM to FT by replacing

$$q(t) \longrightarrow \varphi(x), \tag{2-7}$$

$$p(t) \longrightarrow \pi(x). \tag{2-8}$$

Similarly,

$$\frac{dq(t)}{dt} \equiv \dot{q}(t) \longrightarrow \frac{\partial \varphi(x)}{\partial x^\mu} \equiv \partial_\mu \varphi(x). \tag{2-9}$$

In analogy one can also introduce the Lagrangian,

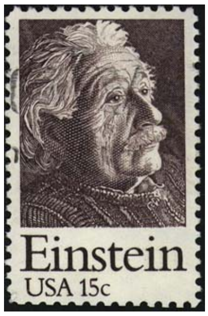
$$L[\varphi,\, \partial_\mu \varphi] = \int d\vec{x}\, \pi(x)\, \partial_0 \varphi(x) - H, \tag{2-10}$$

and give a definition to the conjugate momentum:

$$\pi(x) = \frac{\delta L}{\delta\, \partial_0 \varphi(x)}. \tag{2-11}$$

Furthermore, the minimal action principle, the Euler-Lagrange equations, Noether theorem, and so on can be taken over in a similar way. However, before considering these topics in detail, we need first to understand the properties of fields under coordinate transformations.

2.3 Relativity, Lorentz and Poincaré transformations



The postulate of Einstein's special relativity — the constancy of the speed of light c in all inertial frames — leads to the principle of *Lorentz invariance*: the physics laws must be invariant under Lorentz transformations. Any (successful) attempt of incorporating this principle into the framework of quantum mechanics had led to quantum field theory. Lorentz invariance thus lies in the very heart of QFT.

Consider a light signal emitted along x-direction from the origin of a coordinate frame at rest. After time t, the light signal would arrive at $x = c\,t$. Now let's choose a frame that has the same origin at $t = 0$, but travels with velocity v along the x-axis. If the light speed is constant in all inertial frames, as Einstein requires, then the light arrives at $x' = c\,t'$ in the moving frame (anticipating that time measures differently in the two frames). Now, the coordinates in the two frames should somehow be related. The Galilean laws,

$$t' = t, \quad x' = x - v\,t, \tag{2-12}$$

cannot make it work, but the Lorentz ones can,

$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}, \quad x' = \frac{x - v\,t}{\sqrt{1 - v^2/c^2}}. \tag{2-13}$$

An important property of the Lorentz transformation is linearity, i.e., in general it can be written as

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu\, x^\nu\,. \tag{2-14}$$

It also preserves the length of a space-time interval:

$$\eta_{\mu\nu}x^\mu x^\nu = \eta_{\mu\nu}x'^\mu x'^\nu \,. \tag{2-15}$$

These properties imply that

$$\eta_{\varrho\sigma}\, \Lambda^\varrho{}_\mu \Lambda^\sigma{}_\nu = \eta_{\mu\nu} \,. \tag{2-16}$$

It is often convenient to present Lorentz transformations as 4×4 matrices acting on the 4-vectors:

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \Lambda^0_0 & \Lambda^0_1 & \Lambda^0_2 & \Lambda^0_3 \\ \Lambda^1_0 & \Lambda^1_1 & \Lambda^1_2 & \Lambda^1_3 \\ \Lambda^2_0 & \Lambda^2_1 & \Lambda^2_2 & \Lambda^2_3 \\ \Lambda^3_0 & \Lambda^3_1 & \Lambda^3_2 & \Lambda^3_3 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \tag{2-17}$$

or, in matrix notation:

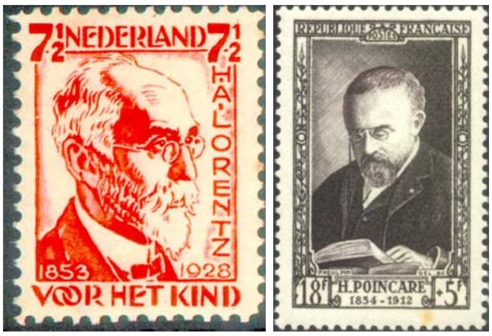
$$x' = \Lambda x \,. \tag{2-18}$$

The consequence of linearity [i.e. Eq. (2-16)] then reads simply as $\Lambda^T \eta \Lambda = \eta$, and since $\eta^2 = 1$ we have:

$$\eta \Lambda^T \eta \Lambda = 1 \,. \tag{2-19}$$

This looks like the orthogonality property, but it's not quite it. Would we be in the Euclidean space ($\eta = 1$), Λ's would be orthogonal. However, for better or worse, we live in Minkowski space, and this make the things they are — not simple.

Nevertheless, just as for orthogonal matrices: $(\det \Lambda)^2 = 1$. Transformations with $\det \Lambda = 1$ and $\Lambda^0_0 \geq 0$ are called *proper*.



In general, Lorentz transformations can be decomposed into: spatial rotations, boosts, and space-time reflections. The reflections are not proper Lorentz transformations.

A more general coordinate transformation involves translations, in addition:

$$x_\mu \to x'_\mu = \Lambda^\mu{}_\nu\, x^\nu + a_\mu \tag{2-20}$$

and is called *Poincaré transformation*, or *inhomogeneous Lorentz transformation*.

In a relativistic theory, the action

$$S = \int d^4x\, \mathscr{L}[\varphi_k(x), \partial_\mu \varphi_k(x)] \tag{2-21}$$

must be invariant under Lorenz transformations, hence: the Lagrangian density $\mathscr{L}$ must be a Lorentz scalar, the field must transform in a Lorenz-covariant fashion, while the solutions of field equations, following from $\delta S = 0$, must, as we shall see, be covariant under the Poincaré transformations. To understand why this is so, it's necessary to dwell a bit into the group theory of continuous transformations.

However, before plunging into a highbrow math, let us see why this is so necessary. Under a coordinate transformation the field will in general transform too:

$$\varphi_k(x) \stackrel{x\to x'}{\to} \varphi'_k(x') = D_{kk'}(\Lambda)\,\varphi_{k'}(\Lambda x), \tag{2-22}$$

where $D(\Lambda)$ is a constant matrix which acts on the discrete indices. When performing two consecutive Lorentz-transformations Λ_1 and Λ_2:

$$\varphi_k(x) \stackrel{\Lambda_1}{\to} D_{kk'}(\Lambda_1)\,\varphi_{k'}(\Lambda_1 \cdot x) \stackrel{\Lambda_2}{\to} D_{kk''}(\Lambda_2)\,D_{k''k'}(\Lambda_1)\,\varphi_{k'}(\Lambda_2\Lambda_1 x), \tag{2-23}$$

we expect that the same result should be achieved in one transformation $\Lambda = \Lambda_1\Lambda_2$. This, obviously, happens only if

$$D_{kk''}(\Lambda_2)\,D_{k''k'}(\Lambda_1) = D_{kk'}(\Lambda). \tag{2-24}$$

Now, if Λ's form a *group*, and they do, then the above equation simply tells us the matrices D must form a *representation* of this group. The fields can then be classified according to the group representation, i.e., according to how it transforms under a Lorentz transformation. All we have to do is to learn about the representations of the Lorentz group.

2.3.1 Lorentz group, algebra, representations

Lorentz transformations Λ form a group, denoted by $\tilde{L}$ and defined by:

1) unit element $\delta^{\mu}{}_{\nu}$, the Kronicker symbol (the 4×4 unit matrix).
2) group element $\Lambda \in \tilde{L}$ and its inverse $\Lambda^{-1} = \eta\Lambda^T\eta \in \tilde{L}$.
3) group property: $\Lambda_1\Lambda_2 = \Lambda_3 \in \tilde{L}$, for any $\Lambda_1,\ \Lambda_2 \in \tilde{L}$.
4) associativity: $(\Lambda_1\Lambda_2)\Lambda_3 = \Lambda_1(\Lambda_2\Lambda_3)$, for any $\Lambda_1,\ \Lambda_2,\ \Lambda_3 \in \tilde{L}$.

This group is very similar to O(4) — the group of orthogonal 4×4 matrices (i.e., group of Lorentz transformations in Euclidean space). However, only the spatial transformations are orthogonal, the boosts are not. The spatial rotations form the group O(3), which is thus a *subgroup* of $\tilde{L}$. Because of these reasons, it is customary to denote the Lorentz group as

O(3,1). The proper Lorentz transformations form a subgroup called the proper Lorentz group and denoted as $\tilde{L}_+$, or SO$^+(3,1)$.

We are interested in *representations* of Lorentz group. As as mentioned earlier, the matrix representations $D(\Lambda)$, are defined by

$$D(\Lambda_1)D(\Lambda_2) = D(\Lambda_1\Lambda_2)\,. \tag{2-25}$$

The trivial representation is given by $D(\Lambda) = 1$. To find some non-trivial ones it is helpful to introduce the infinitesimal transformations, namely: $\Lambda = 1+\omega$, such that $|\omega| \ll 1$. From the property $\Lambda^T\eta\Lambda = \eta$, it follows that $\omega^T = -\omega$, i.e., ω is antisymmetric. In this case $\Lambda_0^0 = 1$, and

$$\det(1+\omega) = e^{\operatorname{tr}\ln(1+\omega)} = e^{\operatorname{tr}\omega} + \mathcal{O}(\omega^2) = 1 + \mathcal{O}(\omega^2), \tag{2-26}$$

therefore we deal with proper transformations only.

Now, the group theory tells us that representations of such a group of infinitesimal transformations can be constructed as

$$D(\Lambda) = \exp\left[\tfrac{i}{2}\operatorname{tr}(\omega M)\right] \tag{2-27}$$

where M is matrix which does not depend on the particular transformation, but represent *generators* of the transformations. Note that M has two sets of matrix indices: one in the space of ω, another in the space of D. The trace is taken in the 4×4 space of ω. To make all this more explicit, we may write the above equation as

$$D_{k'k}(\Lambda) = \exp\left(\tfrac{i}{2}\omega_{\mu\nu}M^{\mu\nu}_{k'k}\right). \tag{2-28}$$

Because ω is antisymmetric, M is antisymmetric in tensor indices μ and ν, hence it has 6 independent elements in this space. It is said that Lorentz group has 6 independent generators. Three of them,

$$J_i = (1/2)\epsilon_{ijk}M^{jk} \tag{2-29}$$

can be identified with generators of spatial rotations, also known as angular momentum operators (corresponding with rotations around the 3 different axes). The remaining three,

$$B_i = M^{0i} \tag{2-30}$$

are called boost generators (corresponding with boosts along the 3 axes).

The most general proper Lorentz transformation can be decomposed into a boost with rapidity $\lambda = \operatorname{arctanh}(v/c)$ along the direction of a unit vector $\vec{u}$ and a rotation over an angle θ around a unit vector $\vec{e}$, with the corresponding representation given by:

$$D(\Lambda) = e^{-i\lambda\,\vec{u}\cdot\vec{B} - i\theta\,\vec{e}\cdot\vec{J}} \tag{2-31}$$

The six generators of the proper Lorentz transformations satisfy the *Lie algebra*:

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk}J_k, \\ [J_i, B_k] &= i\epsilon_{ijk}B_k, \\ [B_i, B_j] &= -i\epsilon_{ijk}J_k. \end{aligned} \tag{2-32}$$

In the first line you recognize the SU(2) algebra, while the rest might not be as familiar. It will help a lot to introduce the following combination of the generators

$$J_k^{\pm} = \frac{1}{2}(J_k \pm iB_k). \tag{2-33}$$

Using (2-32) one can prove that the new generators $J_k^{\pm}$ satisfy the SU(2) algebra too:

$$\begin{aligned} [J_i^{\pm}, J_j^{\pm}] &= i\epsilon_{ijk}J_k^{\pm}, \\ [J_i^{+}, J_j^{-}] &= 0. \end{aligned} \tag{2-34}$$

Only, in this case the generators are complex. That is to say, the algebra of Lorenz generators is SU(2,C) algebra.

Therefore, the representations of the Lorentz group can be obtained from the known representations of SU(2) [or, equivalently, SO(3)]. Since the latter ones are labelled by the spin $s = k + \frac{1}{2}, k$ (with $k \in \mathbb{N}$), any representation of the Lorentz algebra can be identified by specifying (s_+, s_-), the spins of the representations of the two copies of SU(2) that made up the Lorentz algebra (2-32).

The spin-s matrix representation of SU(2) generators can be reconstructed from Clebsch-Gordon coefficients, denoted usually as $C(j_1m_1, j_2m_2; jm)$ or $\langle j_1j_2m_1m_2|j_1j_2jm\rangle$, in the following fashion:

$$\begin{aligned} J_1^{(s)} &= (\sqrt{2})^{-1}\left(\mathcal{C}_{+1} - \mathcal{C}_{-1}\right), \\ J_2^{(s)} &= (i\sqrt{2})^{-1}\left(\mathcal{C}_{+1} + \mathcal{C}_{-1}\right), \\ J_3^{(s)} &= \mathcal{C}_0, \end{aligned} \tag{2-35}$$

where

$$\left(\mathcal{C}_\sigma^{(s)}\right)_{\lambda'+s+1,\,\lambda+s+1} = \sqrt{s(s+1)}\,C(1\sigma, s\lambda;\, s\lambda'), \tag{2-36}$$

with indices running as: $\sigma = \overline{-1,1}$ and $\lambda = \overline{-s,s}$. Note that $J_i^{(s)}$ is a $(2s+1)\times(2s+1)$ matrix. To give an example, $\vec{J}^{(1/2)} = \frac{1}{2}\vec{\sigma}$, where σ_i are the Pauli matrices.

Choosing

$$J_i^{\pm} = J_i^{(s\pm)}, \tag{2-37}$$

Representation	Generators: $-iM^{\mu\nu}$	Field
$(0,0)$	0	$\phi(x)$ — Scalar
$(\frac{1}{2},0)\oplus(0,\frac{1}{2})$	$\frac{1}{4}[\gamma^\mu,\gamma^\nu]_{\alpha\beta}$	$\psi_{(\alpha)}(x)$ — Spinor
$(\frac{1}{2},\frac{1}{2})$	$\delta^\mu_{\ \alpha}\,\delta^\nu_{\ \beta}-\delta^\mu_{\ \beta}\,\delta^\nu_{\ \alpha}$	$A^\alpha(x)$ — Vector
$(1,0)\oplus(0,1)$		$F^{\alpha\beta}(x)$ — Antisymmetric tensor
$(1,1)$		$h^{\alpha\beta}(x)$ — Symmetric traceless tensor
$[(\frac{1}{2},0)\oplus(0,\frac{1}{2})]$ $\otimes(\frac{1}{2},\frac{1}{2})$	$\frac{1}{4}[\gamma^\mu,\gamma^\nu]_{(\alpha\beta)}\eta_{\varrho\sigma}$ $+\,\delta_{(\alpha\beta)}(\delta^\mu_{\ \varrho}\,\delta^\nu_{\ \sigma}-\delta^\mu_{\ \sigma}\,\delta^\nu_{\ \varrho})$	$\psi^\varrho_{(\alpha)}(x)$ — Vector-Spinor

Table 2.1: Some representations of the Lorentz group.

we can reconstruct the (s_+,s_-) representation of the Lorentz generators J_i, B_i, by using Eq. (2-33), and then find $M^{\mu\nu}$ from Eq. (2-29) and (2-30). These are called *finite-dimensional* representations of the Lorentz group. They are not unitary, because the group is non-compact (boosts are not bounded).

Of special interest are symmetric representations, that is (s,s), or $(s,s')\oplus(s',s)$ representations. Several examples are given in Table 2.1.

2.3.2 Poincaré group, algebra, representations

The Poincaré transformations also form a group. In addition to the Lorentz-group generators $M_{\mu\nu}$, it has the generators of translations P_μ, acting on fields as

$$e^{iP_\mu a^\mu}\varphi_k(x)=\varphi_k(x+a)\,. \tag{2-38}$$

The simplest representation for this operator is $P_\mu=i\partial_\mu$.

The generators satisfy the Poincaré algebra:

$$\begin{aligned}[P_\mu,P_\nu] &= 0,\\ [M_{\mu\nu},P_\rho] &= i(\eta_{\nu\rho}P_\mu-\eta_{\mu\rho}P_\nu),\\ [M_{\mu\nu},M_{\rho\sigma}] &= i(\eta_{\mu\sigma}M_{\nu\rho}+\eta_{\nu\rho}M_{\mu\sigma}-\eta_{\mu\rho}M_{\nu\sigma}-\eta_{\nu\sigma}M_{\mu\rho}).\end{aligned} \tag{2-39}$$

Note that the Poincaré generators M differ from Lorentz ones $M^{(L)}$ by an orbital momentum contribution. In the coordinate space representation,

$$M_{\mu\nu} = i(x_\mu \partial_\nu - x_\mu \partial_\nu) + M_{\mu\nu}^{(L)} \tag{2-40}$$

Knowing the algebra is useful for finding the Casimir operators (i.e., operators which commute with all the generators of the group). The Poincaré group has two Casimir operators:

$$C_1 = P_\mu P^\mu, \quad C_2 = W_\mu W^\mu, \tag{2-41}$$

where

$$W^\mu = \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma} M_{\nu\rho} P_\sigma \tag{2-42}$$

is the Pauli-Lubanski vector, with the following properties:

$$\begin{aligned} W^\mu P_\mu &= 0, \\ [W^\mu, P^\nu] &= 0, \\ [W^\mu, M^{\rho\sigma}] &= i(\eta^{\mu\rho} W^\sigma - \eta^{\mu\sigma} W^\rho), \\ [W^\mu, W^\nu] &= -i\epsilon^{\mu\nu\rho\sigma} W_\rho P_\sigma, \end{aligned} \tag{2-43}$$

The group representations can be classified according to the eigenvalues of the Casimir operators. Introducing the states of a“particle” with mass m and spin s, such that mass and spin are conserved under the Poincaré transformations, requires these states to be the eigenstates of the Casimir operators, with the eigenvalues given by:

$$\begin{gathered} P^2 \left|m, s\right\rangle = m^2 \left|m, s\right\rangle, \\ W^2 \left|m, s\right\rangle = -m^2 s(s+1) \left|m, s\right\rangle. \end{gathered} \tag{2-44}$$

It is important to realize that the fields fall into the (finite-dimensional) representations of the proper Lorentz group, while the physical particle states fall into the representations of the Poincaré group. Therefore, the field equations, which are supposed to produce particle states as solutions, must be formulated accordingly: to provide the transition from Lorentz representations to Poincaré representations, and thus to associate fields with particles. Below we consider some examples.

2.4 Free-field equations

2.4.1 Scalar

The *scalar* field $\phi(x)$ transforms as $(0,0)$, which is the trivial representation. Let's starts with the following action for this field

$$S = \int d^4x\, \tfrac{1}{2}(\partial_\mu\phi\, \partial^\mu\phi - m^2c^2\phi^2) \tag{2-45}$$

From the Euler-Lagrange equations one finds The free-field equation is

$$(\partial^2 + m^2c^2)\phi(x) = 0, \tag{2-46}$$

which is of course the Klein-Gordon equation. We can convert this differential equation into an algebraic equation for the Fourier transform:

$$\phi(x) = \int \frac{d^4p}{(2\pi)^4}\Phi(p)e^{ipx}\,. \tag{2-47}$$

As a result we obtain

$$p^2 - m^2c^2 = 0 \tag{2-48}$$

i.e., the mass-shell condition. Note that this equation has two solutions for the energy:

$$E = p^0c = \pm\sqrt{\vec{p}^2c^2 + m^2c^4} \tag{2-49}$$

The Poincaré generators in this case are $P_\mu = i\partial_\mu$, and $M_{\mu\nu} = 0$, therefore the eigenvalues of the Casimir operators are:

$$P^2 = m^2,\ \ W^2 = 0 \tag{2-50}$$

Indeed this equation is suitable for the description of particles with mass m and spin 0.

2.4.2 Dirac field

From now on, we choose units such that $c = 1$.

$$S = \int d^4x\, \tfrac{1}{2}(i\bar{\psi}\,\gamma\cdot\partial\,\psi - m\bar{\psi}\psi) \tag{2-51}$$

$$(i\gamma\cdot\partial - m)\psi = 0 \tag{2-52}$$

hence, again $p^2 = m^2$. The 2nd Casimir operator can be found by substituting the $M^{\mu\nu}$ from the table into the definition of Pauli-Lubansky vector. The result is

$$W^2 = -m^2\tfrac{1}{2}(\tfrac{1}{2} + 1)\,. \tag{2-53}$$

Therefore, this equation describes a particle with mass m and spin 1/2.

2.4.3 Proca field

$$S = \int d^4x \, (- \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{2} m^2 A_\mu A^\mu) \tag{2-54}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

$$\begin{aligned} (\partial^2 + m^2) A^\mu(x) &= 0, \\ \partial_\mu A^\mu &= 0 \, . \end{aligned} \tag{2-55}$$

Note the new feature: an extra equation.

Computing the 1st Casimir operator we find $P^2 = m^2$. The 2nd Casimir operator can again be found by substituting the $M^{\mu\nu}$ from the table into the definition of Pauli-Lubansky vector, with the result:

$$W^2 = -m^2(1 + 1) \, . \tag{2-56}$$

Therefore, this equation describes a particle with mass m and spin 1.

2.5 Literature

- Wu-Ki Tung, *Group Theory in Physics* (Word Scientific, Philadelphia, 1995), chapter 10.
- C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, Singapore, 1988), chapter 2-1.

2.6 Exercises

2.6.1 Poincaré transformations

[4 pts] Demonstrate the group property of the Poincaré transformations.

2.6.2 Representations of the Lorentz group

[8 pts] Using the spin-1/2 representation of SU(2) generators: $J_i^{(1/2)} = \frac{1}{2}\sigma_i$, construct the Lorentz generators $M^{\mu\nu}$ in the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representations. Show that the result can be written as $M^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$, where γ's are Dirac's matrices in the 'chiral representation'.

2.6.3 Rarita-Schwinger field

[2 pts] How does a vector-spinor field $\psi^\mu_{(\alpha)}(x)$ transform under a Lorentz transformation?

[8 pts] Assume the following action for this field (omitting spinor indices):

$$S = \int d^4x\, \bar{\psi}_\mu (i\gamma^{\mu\nu\alpha}\partial_\alpha - m\gamma^{\mu\nu})\psi_\nu \,, \tag{2-57}$$

where $\gamma^{\mu\nu\alpha} = \frac{1}{2}(\gamma^\mu\gamma^\nu\gamma^\alpha - \gamma^\alpha\gamma^\nu\gamma^\mu)$ and $\gamma^{\mu\nu} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$. Show that $\gamma^{\mu\nu\alpha}$ is totally antisymmetric in the vector indices. From the Euler-Lagrange equation, derive the field equation. Show that it gives rise to a Dirac-like equation and constraints.