COMPACT TIME AND DETERMINISM FOR BOSONS:
foundations

Donatello Dolce[*]

Institut für Physik (WA THEP)
Johannes-Gutenberg Universität
D-55099 Mainz
Germany

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Abstract

Free bosonic fields are investigated at a classical level by imposing their characteristic de Broglie periodicities as constraints. In analogy with finite temperature field theory and with extra-dimensional field theories, this compactification naturally leads to a quantized energy spectrum. As a consequence of the relation between periodicity and energy arising from the de Broglie relation, the compactification must be regarded as dynamical and local. The theory, whose fundamental set-up is presented in this paper, turns out to be consistent with special relativity and in particular respects causality. The non trivial classical dynamics of these periodic fields show remarkable overlaps with ordinary quantum field theory interpretable as a generalization of the AdS/CFT correspondence.

[*]e-mail: dolce@thep.physik.uni-mainz.de

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Introduction

Time is a concept that has always played a central role in physics. Its operative definition is given by counting the number of cycles of a phenomenon "supposed" to be periodic. The importance of the assumption of periodicity is also present in the A. Einstein's definition of a relativistic clock

> *"By a clock we understand anything characterized by a phenomenon passing periodically through identical phases so that we must assume, by the principle of sufficient reason, that all that happens in a given period is identical with all that happens in an arbitrary period".*   A. Einstein [1]

As the Einsteinian concept of time itself, the "period" considered in the above definition implicitly has a local and dynamic nature related to the motion of the reference frames. A

manifestation of this aspect is the relativistic Doppler effect, which is a direct consequence of the time interval variations induced by the Lorentz transformations between different reference frames. In fact, in modern physics time emerges from the Minkowski metric as a fourth dimension in addition to the spatial ones.

Despite the great success of special relativity, among the most challenging issues in modern physics there are still those concerning the notion of time. The relativistic laws are compatible with an inversion of the time arrow. On the other hand, from a statistical point of view, the time arrow is fixed by the second law of thermodynamics, which states that the total entropy of the universe must increase for probabilistic reasons. Also in non-relativistic quantum mechanics, time plays a peculiar role with respect to the spatial variables. Within the Hamiltonian formulation, the time dimension emerges dynamically through the Schrödinger equation, whereas the Lagrangian formulation highlights a connection between statistic and quantum mechanics. In fact there is a close analogy between the Boltzmann formulation of statistical mechanics and the Feynman path integral formulation of quantum mechanics [2]. It is well known that the quantization of three-dimensional statistical systems is achieved by adding a periodic time dimension of Euclidean type [3, 4, 5]. This is one of the basic assumptions of field theory at finite temperature. The statistical quantum systems studied in this way are those of quantum fields at thermal equilibrium, whose Euclidean time periodicity is proportional to the inverse of the temperature.

Theories with time periodicity are studied in different branches of physics such as quantum mechanics, thermodynamics, statistical mechanics, cosmology and elementary particle physics. In the canonical formulation of special relativity the space-time dimensions are implicitly assumed to be infinite. However, recently the idea of a compact time dimension at a cosmological scale, *i.e.* a universe intrinsically periodic or cyclic, has become of growing interest for the understanding of the origin and end of the universe [6, 7, 8, 9]. Moreover, in black hole thermodynamics, the space-time curvature due to the black-hole mass $M_\odot$ yields a metric which, upon Wick rotation, gives a time periodicity $8\pi G M_\odot$ and thus the Hawking radiation [3]. Concerning general relativity, we only note that the Einstein equations admit time periodic solutions [10]. We note that the Lorentz transformations emerging from the Minkowskian metric fix the differential structure of special relativity but they do not prescribe any particular restriction to the boundary conditions that must be imposed on the space-time dimensions. On the other hand, the solution of relativistic differential systems requires the assumption of suitable conditions on the four-dimensional boundaries in which the theory is embedded. Generalizing the Hamilton principle, the important requirement is that these conditions minimize the relativistic action on the boundaries, *i.e* they are Hamiltonian constraints [11]. For this reason the fields are usually assumed to have fixed values at initial and final times, however others typologies of boundary conditions such as periodic or antiperiodic ones can be chosen [12].

First we restrict our study to elementary isolated systems, where the Bohr-Sommerfeld quantization condition says that, in a give potential, the allowed phase-space orbits are those with an integer number of periods. This is a periodicity condition. A typical

application of this approach are the Bohr orbitals in the Hydrogen atom. Historically this was one of the first evidences of quantum mechanics.

Continuing using as simple as possible arguments, we note that a naive way to obtain a quantization of the energy is to set its conjugate variable, namely the physical time, on a finite interval. This is in close analogy with the Matsubara theory [13] where the assumption of a periodicity condition in the Euclidean time yields a discretized thermal energy tower whose levels are known as Matsubara energies. Another similitude is given by the Kaluza-Klein theory [14, 15] where a discrete mass spectrum is obtained by imposing that the field is embedded in a compact spatial extra-dimension. Since the proper time is the conjugate variable of the invariant mass, we will address it as "virtual" extra dimension. Provided the identification of the extra coordinate with the time is made, in extra dimensional theories the determination of the mass spectrum is obtained from a differential system analogous to the Schrödinger one [16, 17, 18]. This is a consequence of the fact that the Klein-Gordon equation is the relativistic generalization of the Schrödinger equation.

The key assumption of this work is a generalization of the de Broglie hypothesis [19, 20]: every field has a given angular frequency $\bar{\omega}$ (as long as it does not interact); the energy of the related quanta $\bar{E} = \hbar\bar{\omega} = h/T_t$ is fixed by the inverse of the period T_t through the Planck constant h. We will impose such de Broglie periodicities T_t as constraints to the fields and we will obtain a quantized tower of energy resonances with gap $\bar{E} = h/T_t$. The energy resonances associated to such a T_t periodic field will be interpreted in terms of quanta with energy $\bar{E}$. To obtain a consistent relativistic invariant theory we will of course also consider the periodicities induced by the time dimension on the modulo of the spatial dimensions and, for massive fields, on the proper time. To see this consistence it is important to bear in mind that we will always impose the usual de Broglie space-time periodicities of the relativistic fields as constraints.

To generalize the "old" quantum theory we will focus on periodic scalar fields, *i.e.* packets of free relativistic waves satisfying the same periodic boundary conditions. Such a harmonic system, similar to acoustic waves, is one of the simplest and most fundamental systems in nature. By considering compact space-time dimensions, the periodic fields will be described by a quantized energy-momentum spectrum. The main difference with to Kaluza-Klein theory is that the compactification periodicities are now fixed by a dynamical parameters like the energy or the momentum and not by an invariant parameter like the mass. Therefore such a compactification must be regarded as dynamical and local, and not statical and invariant as in the Kaluza-Klein model. In this way, sec.(1), we will find for both massless and massive fields that every energy eigenmode has the correct relativistic dispersion relation, so that the quantization that we obtain corresponds to the normal ordered energy spectrum of the ordinary quantum relativistic fields. In particular the mass is given by the inverse of the proper time ("virtual" extra-dimension) periodicity which in turn fixes the upper limit of the physical time periodicity through Lorentz transformations. This intrinsic time periodicity is know as de Broglie periodic phenomenon or de Broglie internal clock of massive particles. A hypothetic boson with the mass of an electron has an intrinsic rest periodicity, proportional to the Compton wave length, of about $10^{-20}s$.

Even for a mass as light as that, the periodic dynamics are extremely fast. For instance, the oscillation period of the Cs-133 atom, which is used in the operative definition of time, is of the order of $10^{-10}s$. Remarkably, this intrinsic periodicity of massive particles has been indirectly observed only in a recent experiment [21] for electrons (which in our view can be though of as fields with antiperiodic T_t).

The theory is based upon relativistic waves with the boundary conditions that minimize the relativistic action, thus every perturbation in a given point propagates with the retarded and advanced potential - as well as the information. The resulting periodicities for these fields are indeed dynamical and local. This means that the compactification radius must *not* be regarded as static since it changes according to the relativistic causality and to energy conservation. In other words, interactions destroy the original periodicity so that the system passes from a periodic regime to another periodic regime depending on the amount of energy exchanged, just as in Compton scattering. In this way we can order events in time and we will conclude that the dynamical compactification arising from our theory respects all the fundamental requirements for a well formulated notion of relativistic time.

To have a simple image of our assumption one should remember how acoustic waves are described in terms of objects vibrating within compact spatial dimensions. In a full relativistic generalization of acoustic waves, our quantum-relativistic fields can be regarded as imbedded in compact space-time dimensions. Thus we want to consider all the harmonics modes allowed by the de Broglie periodicities, not only the fundamental ones of the usual approach. Ordinary field theory, which is supposed to describe every elementary systems, is based upon de Broglie waves whose characteristic periods can be regarded as internal (de Broglie) clocks. The usual relativistic time axis is defined by reference to the "ticks" of these periodic phenomena, in particular to the ones of the Cs-133 atomic clock. For massless (electromagnetic or gravitational) fields these periodicities can be in principle infinite. In general the periodicities vary through energy exchange and the combination of two periodic systems results in ergodic (not exactly periodic) evolution. Every value of the time axis is characterized by a combination of the different phases of these de Broglie internal clocks of the elementary fields considered. Hence the external time axis can be dropped. Considering the Einstein's definition of relativistic clock we notice that the physical information for the fields is in the single periods. This leads to dynamic compact intervals with periodic conditions.

The aim of this investigation is to stress the analogy between dynamic periodic fields and the usual quantum fields, sec.(2). From the analogy with the Kaluza-Klein theory we will find that the energy eigenstates has a Marcovian time evolution which is described by the Schrödinger equation. Being stationary waves, they form a complete set with an underlying inner product which can be used to build a Hilbert space. Formally, with this at hand, the Feynman path integral for free bosonic fields arises without any further assumptions. Due to the periodic nature of the fields we will be able to extract the commutation relations as well as other aspects of quantum mechanics such as the Heisenberg relation and a generalization of the Bohr-Sommerfeld condition.

Since quantum behaviors arise from a classical system we can talk about determinism or pre-quantization. A proposal for the possible deterministic nature of quantum theory has been given by 't Hooft who [22, 23, 24] had shown that there is a close relation between a classical particle moving on a circle and the quantum harmonic oscillator. This model can be thought of as having a periodic time on a lattice, [25, 26]. Similarly to a rolling dice, if the periodicity is too fast, at every observation the system results in a different phase of an apparently aleatoric evolution [27]. The system can only be described by using a transition probability from one state to another which turns out to be in agreement with the usual quantum rules for the harmonic oscillator. Within these scenarios important efforts to formulate quantum mechanics from a classical theory with compact extra-dimensions have been made in [28, 29, 30]. In this case ergodic dynamics reproduce quantum behaviors in terms of an emerging effective time. It is important to note that, in our theory, by supposing that the quantum behavior arises from periodicity boundary conditions, we are avoiding the introduction of hidden variables and at the same time we are implicitly introducing a non-locality, so that our model is not constrained by Bell's theorem [31].

1. Dynamic approach to compactified time

The differential structure of relativistic kinematics is based on the four-dimensional Minkowski metric

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - d\mathbf{x}^2 \ , \tag{1}$$

and the related Lorentz transformations. In this metric a Klein-Gordon complex field $\Phi_{KG}(\mathbf{x}, t)$ with mass M obeys the equation $(\partial_\mu \partial^\mu + M^2 c^2/\hbar^2)\Phi_{KG}(\mathbf{x}, t) = 0$ and it holds the relativistic dispersion relation $E^2(\mathbf{p}) = |\mathbf{p}|^2 c^2 + M^2 c^4$. It is worth noting that the solutions of this differential system depend on the boundary conditions imposed on the field and that neither the Minkowskian metric nor the Lorentz transformations prescribe restrictions to them. In general, boundary conditions must be chosen such as to be consistent with the variational principle applied on the boundaries [11]. For simplicity in this preliminary discussion we will concern only with time dimension boundaries. Thus, for a generic scalar field $\Phi(\mathbf{x}, t)$ with time evolution inside the an interval $t \in [t', t' + 2\pi R_t]$ and no additional boundary terms, the important requirement is that

$$\int d^3 x \left[\delta\Phi(\mathbf{x}, t)\partial_t \Phi(\mathbf{x}, t)\right]_{t'}^{t' + 2\pi R_t} \equiv 0 \ . \tag{2}$$

In ordinary field theory this relation is satisfied by choosing fixed values of the field at the initial and final times $\delta\Phi(\mathbf{x}, t') = \delta\Phi(\mathbf{x}, t' + 2\pi R_t) \equiv 0$.[1] However also periodic, antiperiodic or (more generally) combinations of Dirichlet ($\Phi = 0$) and Neumann ($\partial_t \Phi = 0$) boundary conditions are compatible with the variational principle [12]. From eq.(2) we see that these conditions have the same formal validity of the usual conditions assumed in ordinary relativistic field theory - they act as Hamiltonian constraints.

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[1]The condition eq.(2) is invariant under time translations.

In particular, we want to explore at a classical level the physics of a free scalar field $\Phi(\mathbf{x}, t)$ imposing periodicity as a constraint, which means the following invariance

$$\Phi(\mathbf{x}, t') \equiv \Phi(\mathbf{x}, t' + 2\pi R_t) \ . \tag{3}$$

Imposing periodicity along the physical time dimension in order to satisfy eq.(2) and to fix a particular solution of the field equation, doesn't necessarily mean to localize the field $\Phi(\mathbf{x}, t)$ in a particular space-time region. Skipping mathematical details, an equivalent way to interpret the condition eq.(3) is to take the time either on a compact interval $t \in [t', t' + 2\pi R_t]$ or on the whole interval in $\mathbb{R}$ where periodic condition eq.(3) is supposed to be satisfied. In both cases the whole physical information of is contained in a single period and we can restrict our analysis to this region. Using a terminology common in Kaluza-Klein theories we can say that the time dimension is compactified on a circle $t \in \mathbb{S}^1_{R_t}$ with compactification radius R_t.[2]

Provided analogous periodic conditions along the spatial and, for massive fields, proper time dimensions such as to guarantee covariance, we shall see that this theory of periodic fields is consistent with special relativity. In other words we want to impose the natural (de Broglie) periodicities of the relativistic fields as a constraints to determine uniquely the solution of the Klein-Gordon differential equation in every space-time point. In this way we want to generalize the "old" formulation of quantum mechanics: free bosonic fields are supposed to have intrinsic periodicities T_t, so that the energy of the related quanta $\bar{E} = \hbar\bar{\omega} = h/T_t$ depends on the inverse of the time period T_t. This assumption can be regarded as the combination of the Newton's law of inertia with the de Broglie hypothesis.

1.1. Massless bosonic fields

Relativistic massless fields with time periodicity T_t imposed as a constraint are described by the following massless Klein-Gordon action

$$\mathcal{S}[T_t] = \frac{1}{2} \int_0^{\lambda_x} d^3x \int_{t'}^{t'+T_t} dt \left[\partial_\mu \Phi^*(\mathbf{x}, t)\partial^\mu \Phi(\mathbf{x}, t)\right] \ . \tag{4}$$

To satisfy the variational principle on the time boundaries eq.(2) we impose the periodic condition eq.(3). Thus we can decompose the field in eigenmodes

$$\Phi(\mathbf{x}, t) = \sum_{n=-\infty}^{\infty} \Phi_n(\mathbf{x})u_n(t) \ , \tag{5}$$

where the time evolutions are given by $u_n(t) = \exp[-i\omega_n t]$ and the angular frequencies must be $\omega_n = n/R_t$. They are in fact the harmonic modes of a string with periodic boundary conditions. The eigenfunctions $u_n(t)$ form a complete and orthogonal set, so that we can decompactify the action along the time dimension

$$\mathcal{S}[T_t] = \frac{T_t}{2} \int_0^{\lambda_x} d^3x \sum_n \left[\partial_i \Phi_n^*(\mathbf{x})\partial^i \Phi_n(\mathbf{x}) + \frac{\omega_n^2}{c^2}|\Phi_n(\mathbf{x})|^2\right] \ , \tag{6}$$

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[2]This does not mean to deform the flat Minkowskian background eq.(1).

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obtaining a sum over three-dimensional actions of the eigenmodes $\Phi_n(\mathbf{x})$.

Assuming the de Broglie relation, we proceed similarly to the Kaluza-Klein theory or to the Matzubara theory and we associate the quantized frequency spectrum ω_n of the n-th eigenmode to quantized energy spectrum E_n, where the proportionality constant is the reduced Planck constant $\hbar$. In fact, calling the wave number for the n-th eigenmode $\mathbf{k}_n$, we find

$$\Box\Phi(x,t) \equiv 0 \rightarrow \left(\frac{n}{R_t}\right)^2 - \mathbf{k}_n^2 c^2 = 0 \ . \tag{7}$$

Comparing with the massless dispersion relation $E^2 - |\mathbf{p}|^2 c^2 = 0$, it is natural to assume

$$E_n \equiv \hbar\omega_n = \frac{n\hbar}{R_t} \ . \tag{8}$$

After the decompactification we have a tower of energy eigenstates exactly as in extra-dimensional theories one finds a tower of massive eigenstates. The main difference with extra-dimensional theories is that the mass (and thus the compactification radius of the extra-dimension) is four-dimensional invariant (but not five-dimensional invariant) whereas the energy of the field is a dynamical quantity. As we will discuss in details, the time compactification radius must be therefore regarded as dynamical. Here it is worth to note that every mode has a positive defined energy as the Kaluza-Klein modes have always positive defined masses (no tachyonic modes).

To obtain a consistent relativistic theory we must consider also the compactification of the spatial dimensions. In particular, since we are assuming massless on-shell fields, we find that the time periodicity induces a periodicity on the absolute values of the spatial dimensions as well. From eq.(7) it follows that the absolute values of the momenta must be discretized

$$|\mathbf{p}_n| = \hbar|\mathbf{k}_n| = \frac{n\hbar}{R_t c} \ . \tag{9}$$

In other words, since in the massless case the field is on the light-cone $ds^2 = 0$, we get $c^2 dt^2 = d\mathbf{x}^2$ and thus an induced spatial periodicity

$$\lambda_x = T_t c = \frac{nh}{|\mathbf{p}_n|} \ . \tag{10}$$

The gap between the energy levels can be expressed in terms of the energy of the fundamental level (the energy of the eigenmode with $n = 1$). We denote the quantities related to this fundamental level with the bar sign, so it has energy, momentum and angular frequency $\bar{E}$, $\bar{\mathbf{p}}$ and $\bar{\omega}$ respectively. Hence, eq.(8), the compactification radius is fixed by the fundamental energy through the following relation

$$R_t(\bar{\mathbf{p}}) \equiv \frac{\hbar}{\bar{E}(\bar{\mathbf{p}})} = \frac{1}{\bar{\omega}(\bar{\mathbf{p}})} \ . \tag{11}$$

This relation (namely the de Broglie relation) emerges naturally from the periodic field formulation and we take it as one of the basic assumption of this work. In fact it
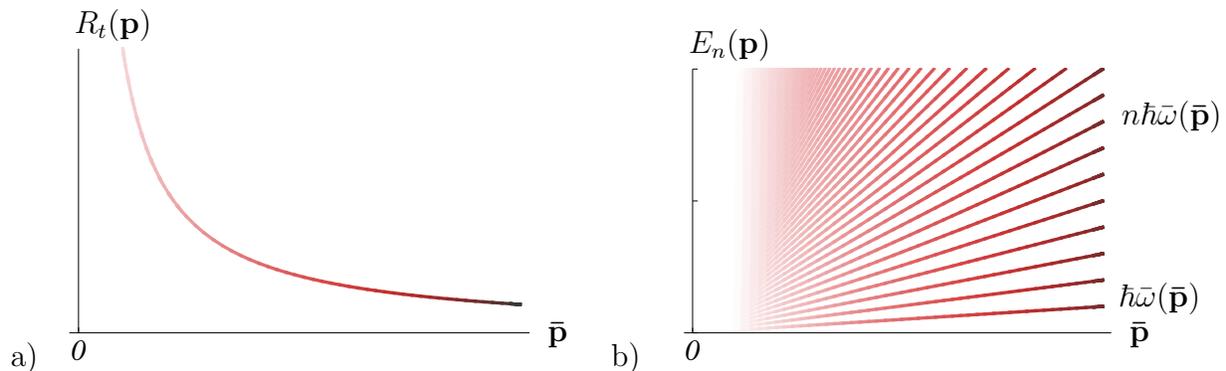
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Figure 1: Spectral behavior for a massless periodic field as a function of the fundamental momentum $\bar{\mathbf{p}}$. Fig.(a) shows the variation of the compactification radius $R_t(\bar{\mathbf{p}}) \equiv \hbar/\bar{E}(\bar{\mathbf{p}})$, according to $\bar{E}(\bar{\mathbf{p}}) = \hbar\bar{\omega}(\bar{\mathbf{p}}) = |\bar{\mathbf{p}}|c$. Fig.(b) shows the massless relativistic dispersion relation of the resulting energy spectrum $E_n(\bar{\mathbf{p}}) = n\hbar\bar{\omega}(\bar{\mathbf{p}})$. In the limit of zero momentum the fundamental compactification radius tends to infinity giving a continuos energy spectrum.

has a geometrical interpretation in terms of the compactification length (compression) of the vibrating string. The dispersion relation for the first eigenmode in this massless case, eq.(9), is

$$\bar{\omega}(\bar{\mathbf{p}}) = \frac{|\bar{\mathbf{p}}|c}{\hbar} \ . \tag{12}$$

The relevant aspect of this result is that there is a discretized (quantized) energy spectrum and the Planck constant h relates the temporal period to the inverse of the energy. From these considerations we finally check that the four-momentum of the fundamental level and the space-time compactification radiuses can be written respectively as $\bar{p}_\mu = (\bar{E}/c, \bar{\mathbf{p}})$ and $R_\mu = (cR_t, \mathbf{R}_x)$, where $|\mathbf{R}_x| = R_x = \hbar/|\bar{\mathbf{p}}|$. Generalizing the de Broglie hypothesis, the fundamental compactification conditions eq.(11) and eq.(10) can be written with the following notation

$$R_\mu = \frac{\hbar}{\bar{p}^\mu} \ . \tag{13}$$

From eq.(11) and eq.(12) we see that the usual relativistic massless field is obtained in the limit of infinite compactification radius, or equivalently by taking the radius constant and doing the limit of small $\hbar$, see fig(1.a). Both limits tend to decrease the gap between the energy levels, thus obtaining a continuous energy spectrum, as shown on the left side of fig(1.b). On the contrary, in the limit of large $\hbar$ or high fundamental frequency $\bar{\omega}(\bar{\mathbf{p}})$, we obtain a well discretized energy spectrum, right side of fig(1.b). For the energy levels of periodic fields at the thermal equilibrium it is natural to assume the Boltzmann occupation probability $\propto \exp[-n\bar{E}/K\mathcal{T}]$, where $\mathcal{T}$ is the temperature of the thermal bath. As explained in details in [32], if $K\mathcal{T} \gg \hbar\bar{\omega}$ many energy levels are populated and the field can be approximated by a continuos energy spectrum. On the other hand, if $K\mathcal{T} \ll \hbar\bar{\omega}$, only few levels are populated; here the quantization of the energy spectrum

is manifest. This is the condition needed to avoid the UV catastrophe in the black body radiation or the single photon limit.

1.2. Massive bosonic fields

The key assumption for a massive relativistic field is that it is possible to choose a reference system (the rest frame) where the real time and the proper time can be identified.[3] Therefore we must consider that the compactification of the real time induces a compactification of the proper time, as well.

We approach the theory as a Kaluza-Klein theory for a massless five-dimensional Klein-Gordon field with periodic extra-dimension s and periodic real time. In fact the resulting five-dimensional metric is $dS^2 = c^2 dt^2 - d\mathbf{x}^2 - ds^2 \equiv 0$, so that assuming $s = c\tau$ we recover the usual four-dimensional Minkowskian metric eq.(1). For this reason we will say that the proper time τ acts like a "virtual" extra-dimension[4] whose length is therefore fixed by the time periodicity in the rest frame. We temporarily write the scalar field as a double sum over eigenstates, one over discrete energies because of the periodic time and one over discrete masses because of the induced periodicity on the proper time

$$\Phi(\mathbf{x}, t, s) = \sum_{n_t, n_s} e^{-in_t \bar{\omega} t + in_s \bar{\sigma} s} \Phi_{n_t, n_s}(\mathbf{x}) \ , \tag{14}$$

where $\bar{\sigma} = 2\pi/\lambda_s$ and $\lambda_s = cT_\tau$. The (virtual) five-dimensional Klein-Gordon action is

$$\mathcal{S}[T_t] \ = \ \frac{1}{2\lambda_s} \int_0^{\lambda_s} ds \int_0^{T_t} dt \int_0^{\lambda_x} d^3x \left[\partial_M \Phi^*(\mathbf{x}, t, s) \partial^M \Phi(\mathbf{x}, t, s) \right] \ . \tag{15}$$

Decompactifying the proper time, in analogy with eq.(6) and eq.(11), we have obtained a tower of (virtual) four-dimensional Kaluza-Klein fields with invariant mass gap

$$\bar{M}c^2 \equiv c\hbar\bar{\sigma} = \frac{\hbar}{R_\tau} \ . \tag{16}$$

Then we decompactify the real time obtaining a double tower of three-dimensional eigenmodes $\Phi_{n_s, n_t}(\mathbf{x})$ which satisfy the dispersion relation

$$n_t^2 \frac{\bar{\omega}^2}{c^2} = \mathbf{k}_{n_s, n_t}^2 + n_s^2 \frac{\bar{M}^2 c^2}{\hbar^2} \ . \tag{17}$$

As it is evident at constant spatial separation ($d\mathbf{x}^2 = 0$) or equivalently at zero momentum ($\mathbf{k}_{n_s, n_t} \to 0$), so that the proper time and real time periodicities are identified ($d\tau^2 = dt^2$), we obtain the condition $n_s = n_t = n$. In fact, there is a single periodicity which is

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[3] For sake of simplicity we address: the invariant time as the proper time, the euclidian time as the imaginary time, and the Minkowskian time as the real or physical time.

[4] The Kaluza-Klein quantized mass spectrum can be regarded as a consequence of the fact that the extra-dimension acts similarly to a proper time. In fact, in field theory the conjugate variable of the proper time is the mass and therefore, by putting the proper time in a compact segment we have a quantization of the mass spectrum, that is of the rest frame energy spectrum.
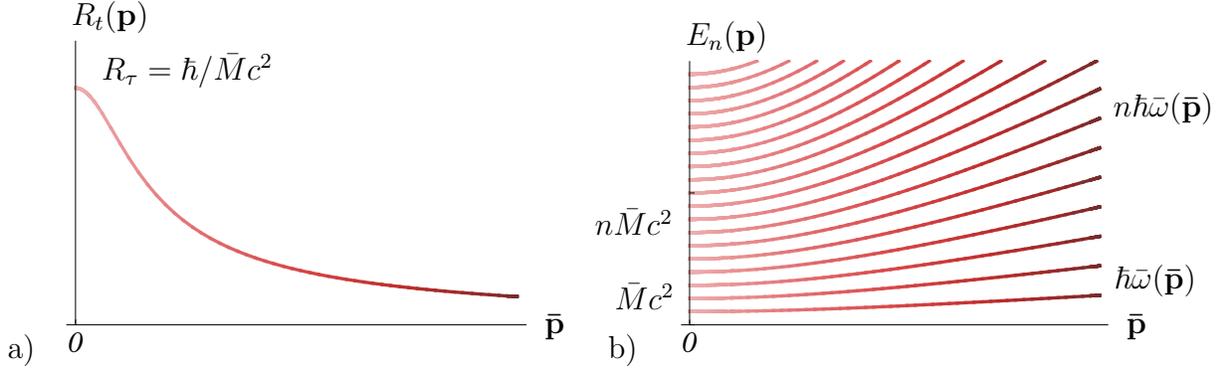
Figure 2: Spectral behavior for a massive periodic field with mass $\bar{M} = \hbar/c^2 R_\tau$ as a function of the fundamental momentum $\bar{\mathbf{p}}$. In fig.(a) it is shown the variation of the compactification radius $R_t(\bar{\mathbf{p}}) \equiv \hbar/\bar{E}(\bar{\mathbf{p}})$, according to $\bar{E}(\bar{\mathbf{p}}) = \hbar\bar{\omega}(\bar{\mathbf{p}}) = (\bar{\mathbf{p}}^2 c^2 + \bar{M}^2 c^4)^{1/2}$. In fig.(b) shows the massive relativistic dispersion relation of the resulting energy spectrum $E_n(\bar{\mathbf{p}}) = n\hbar\bar{\omega}(\bar{\mathbf{p}})$. The proper time compactification radius R_τ fixes the upper limit for $R_t(\bar{\mathbf{p}})$.

induced to the other dimensions and the final result must be a single sum over eigenmodes. Finally from eq.(17) we obtain

$$\bar{\omega}(\bar{\mathbf{p}}) = \frac{\sqrt{\bar{\mathbf{p}}^2 c^2 + \bar{M}^2 c^4}}{\hbar} \ , \tag{18}$$

where $\hbar \mathbf{k}_{n_s,n_t} = n\bar{\mathbf{p}}$. Thus $E_n(\bar{\mathbf{p}}) = n\hbar\bar{\omega}(\bar{\mathbf{p}}) = n\sqrt{\bar{\mathbf{p}}^2 c^2 + \bar{M}^2 c^4}$. As in the massless case eq.(10), the induced spatial periodicity is $\lambda_x = h/|\bar{\mathbf{p}}|$. The action eq.(15) can be seen as a sum over energy eigenmodes with masses $M_n = n\bar{M}$, similarly to eq.(6),[5] [6]

$$\mathcal{S}[T_t] = \frac{T_t}{2} \int_0^{\lambda_x} d^3 x \sum_n \left[n^2 \frac{\bar{\omega}^2}{c^2} |\Phi_n(\mathbf{x})|^2 - |\partial_i \Phi_n(\mathbf{x})|^2 - n^2 \frac{\bar{M}^2 c^2}{\hbar^2} |\Phi_n(\mathbf{x})|^2 \right] \ . \tag{19}$$

We have indeed obtained that the discretized energy spectrum in terms of the compactification radius is still given by eq.(11), but now the eigenstates obey the dispersion relation of relativistic massive particles, as shown in fig(2.b). We note that this quantization is exactly the same one obtained from the usual normal ordered second quantization.

From eq.(16) we can interpret the mass as the inverse of the proper time periodicity: the bigger the mass the shorter the proper time period. Since the energy is bounded from below by the mass $\bar{E}(\bar{\mathbf{p}}) \geqslant \bar{M}c^2$, the time compactification radius has the upper invariant bound $R_t(\bar{\mathbf{p}}) \leqslant R_t(0) = R_\tau$. Roughly speaking we can actually say that the mass fixes the inertia of the motion, fig(2.a). This proper time periodicity of the field, known as the de Broglie periodic phenomenon or the de Broglie internal clock of massive particles

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[5]Note that the periodic massive field obtained in this section is not properly the Klein-Gordon field. The mass term arises by compactification.

[6]Extending our terminology we could say that the energy eigenmodes of the towers are "virtual" Kaluza-Klein particle.

[19, 20, 33], is

$$T_\tau \equiv T_t(0) = \frac{h}{\bar{M}c^2} \; .$$ (20)

To this periodicity the invariant length

$$\lambda_s \equiv T_\tau c = \frac{h}{\bar{M}c}$$ (21)

is associated, which is nothing else than the Compton wave length. A hypothetical light boson for instance with the mass of an electron has a Compton wave length $\lambda_s \sim 2\pi \times 4 \cdot 10^{-13} m$ which leads to the proper time periodicity $T_\tau \sim 8 \cdot 10^{-21} s$. Even for such a light particle, this microscopic time scale can not yet be observed directly in the modern experiments.[7] However, as shown in a recent experiment [21], the modern techniques are reaching a sufficient precision to allow indirect evidences of the de Broglie internal clock of the electron - the proper time periodicity of eq.(20) must be regarded as a general property for massive particles.

A massive periodic field turns-out to be localized inside the Compton wavelength. Furthermore [32], the non-relativistic limit corresponds to a low intensity $|\bar{\mathbf{p}}| \ll \bar{M}c$ massive field where only the first energy level is largely populated and we get the usual non-relativistic free particle distribution (modulo the de broglie internal clock) $\phi(\mathbf{x}) \sim \exp\left[-i\frac{\bar{M}c^2}{\hbar}t + i\frac{\bar{M}}{\hbar}\frac{\mathbf{x}^2}{2t}\right]$. This gives a consistent interpretation of the dualism between waves and particles and also of the double slit experiment.

1.3. Lorentz transformations and covariance

To see that the periodicity in physical time and in proper time are consistent with special relativity, we perform a Lorentz transformation $R_\tau = \gamma(R_t - \mathbf{v} \cdot \mathbf{R}_x/c^2)$ with $\gamma = 1/\sqrt{1-\mathbf{v}^2/c^2}$, from the rest frame of the massive field to another reference frame at velocity $\mathbf{v}(\bar{\mathbf{p}})$. We find that [33] the relation $\hbar = \bar{M}c^2 R_\tau = E(\bar{\mathbf{p}})R_t - \bar{\mathbf{p}} \cdot \mathbf{R}_x$ is perfectly in agreement with the behavior of the periodicity obtained in eq.(18), see also fig.(1.a) and fig.(2.a). This shows why the time periodicity emerges as a dynamical constraint. It is different if observed from different reference systems, exactly as every other time interval in special relativity. Generalising the notation of the massless case, R_μ is dynamically fixed by the four-momentum $\bar{p}_\mu$ as in the de Broglie hypothesis, see eq.(13). In fact, the four-dimensional wavevector $k_\mu = (\bar{\omega}(\bar{\mathbf{p}})/c, \bar{\mathbf{k}})$ is Lorentz-covariant, whereas $\bar{k}_\mu R^\mu$,[8] being a phase of the relativistic fields, is invariant under Lorentz transformations [19, 20, 33].

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[7]It is worthwhile noting that the operative definition of time is given concretely by counting the number of periods of a "well known periodic system". The most accurate definition of second is the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom [SI]. By definition this period is $\sim 10^{-10} s$, whereas the best experimental resolution on resolving time known to the author is about $\sim 10^{-16} s$ [34]. On the other hand, the periodicity explored at the TeV scale is of the order of $\sim 10^{-27} s$.

[8]Since R_μ is such that $\exp[-i\bar{p}^\mu x_\mu] \equiv \exp[-i\bar{p}^\mu(x_\mu + 2\pi R_\mu)]$, it turns out to be a contravariant four-vector.

In the massive case the space-time compactification radiuses can be used to write the relativistic dispersion relation as $\bar{M}^2 c^4 = c^2 \bar{p}^\mu \bar{p}_\mu = (c\hbar/R_\mu)(c\hbar/R^\mu)$. Thus we find $R_\tau^{-2} = R_t^{-2}(\bar{\mathbf{p}}) - R_x^{-2}(\bar{\mathbf{p}})$, so that we can associate to the mass $\bar{M}$, to the energy $\bar{E}$ and to the momentum $\bar{\mathbf{p}}$ a geometrical interpretation in terms of the compactification radius of the proper time R_τ^2, of the real time R_t^2 and of the spatial coordinates $\mathbf{R}_x$, respectively. In few words, since the periodicities that we are imposing in eq.(13) are nothing else than the de Broglie periodicities, the model turns out to be automatically consistent with special relativity.

At this point a small digression about Lorentz invariance in theories with compact dimensions is in order. Through the decompactification of the time dimension in eq.(4) we obtain an equivalent theory with an infinite sum of three-dimensional eigenmodes eq.(6). The equivalence means that this infinite sum over three-dimensional modes is four-dimensional Lorentz invariant as the original formulation. In general, the Lorentz invariance breaking is not because of the compactness of a dimension but rather because that, in an effective lower dimensional theory, only a finite number of eigenmodes can be considered. By generalizing the Higgsless gauge symmetry breaking mechanism induced by boundary conditions as in extra-dimensional theories [12] to a Yang-Mills theory with compact time, it is possible to show [32] that there is a quantization of the magnetic flux and other characteristic behaviors typical of the superconducting regime [35].

1.4. Retarded potential and causality

Taking for simplicity only propagation of massless fields, eq.(4) is sufficient to fix the relativistic Green function. The retarded or advanced Green function $G_{ret(adv)}(\mathbf{x}, t; \mathbf{x}', t')$ is formally the solution of the inhomogeneous relativistic wave equation with point-like source in $(\mathbf{x}', t')$. The Kirchhoff formulation allows us to write the solution for the field $\Phi(\mathbf{x}, t)$ as a source term plus boundary terms[9] at generics initial and final time t_1 and t_2 [2]

$$
\begin{aligned}
\Phi(\mathbf{x}, t) \;=\; & \int_{t_1}^{t_2} dt' \int_{-\infty}^{\infty} d^3 x' G(\mathbf{x}, t; \mathbf{x}', t') j(\mathbf{x}', t') \\
& - \frac{1}{4\pi c^2} \int_{-\infty}^{\infty} d^3 x' \left[G(\mathbf{x}, t; \mathbf{x}', t') \partial_{t'} \Phi(\mathbf{x}', t') - \Phi(\mathbf{x}', t') \partial_{t'} G(\mathbf{x}, t; \mathbf{x}', t') \right]_{t_1}^{t_2}. \quad (22)
\end{aligned}
$$

As we can see from this equation, a variation of the source term induces a retarded variation on the periodicity of the field which must therefore be regarded as dynamical in the sense that it can vary through interactions. In fact, only null sources $j(\mathbf{x}, t)$, or eventually sources with the same periodicity of the field, are compatible with static space-time periodicities of the field itself.[10] In the boundary terms the field acts similarly to a source term and the variation of periodicity of the field propagates in agreement

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[9]As in the introductory discussion about periodic boundary conditions, we are not considering explicitly the spatial boundary terms.

[10]This particular case leads to the so called billiard Green function or similar Green functions that must be not confused with the thermal Green function.

with relativistic causality. This aspect is related to the dynamic and local nature of the compactification already discussed eq.(11) and can be interpreted in terms of the HuygensFresnel principle.

At this point it is important to note that the theory is based upon relativistic waves. Thus the information propagates in agreement with the relativistic causality. By energy conservation, a source term turned on in a given space-time point changes the energy in another space-time point after a time delay, according to eq.(22). Therefore, assuming the dynamical compactification as in eq.(11) and energy conservation, we observe that when the interaction is turned on, together with the energy irradiated, there is an induced variation of the compactification radius from which more complicated and general time evolutions can be obtained. For instance, in fig.(3) we suppose to turn on a source with time periodicity $T_t^\gamma(\bar{\mathbf{p}}_\gamma)$ in the origin of the axis, so that a massless field is "absorbed" after a time delay by a massive field originally at rest. The scenario is similar to the Compton scattering but the energy conservation among the quanta can now be written as a conservation of the inverse of the periodicities between the fields involved in the interaction: $1/T_t^\gamma(\bar{\mathbf{p}}_\gamma) + 1/T_t^{pt}(0) = 1/T_t^{pt}(\bar{\mathbf{p}}')$.

Figure 3: Causality for dynamically periodic fields. A massive field at rest in $|\mathbf{x}| = d$ with periodicity $T_t^{pt}(0)$ "absorbs" after a delay $t = d/c$ the energy of a massless field with periodicity $T_t^\gamma(\bar{\mathbf{p}}_\gamma)$. After the interaction the resulting periodicity $T_t^{pt}(\bar{\mathbf{p}}')$ of the massive field is such that $1/T_t^\gamma(\bar{\mathbf{p}}_\gamma) + 1/T_t^{pt}(0) = 1/T_t^{pt}(\bar{\mathbf{p}}')$ due to energy conservation in the interaction point.

Through interaction, the field passes from the original periodic regime to a different one. Contrary to the static compactification scenario, this means that we can distinguish between an interaction before and an interaction after absorption. Thus we can give a time order to events and a dynamical compactification of the time dimension is compatible with relativistic causality. The theory is therefore in agreement with special relativity and the notion of time is formally well defined. This remarkable result can be interpreted as a consequence of the fact that periodic (or antiperiodic) boundary conditions satisfy the variational principle, exactly as the usual boundary conditions with fixed values of the field, see eq.(2). As the Newton's law of inertial doesn't imply that non-isolated particles go on a straight line, our assumption of periodicity doesn't imply that a system of interacting elementary fields should appear to be periodic.

2.   Periodic mechanics

The previous results encourage an analysis of the mechanics of such periodic fields which we expect to be non trivial due to self-interference. The theory so far is analogous to finite
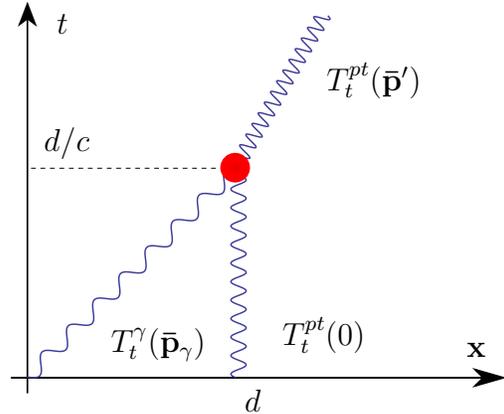
temperature field theory - with Minkowskian compact time - so it yields to a well defined field theory. For both massless and massive periodic fields we can explicitly write down the on-shell solutions of the equations of motion of the actions eq.(6) or eq.(19) with the following notation [5]

$$\Phi(\mathbf{x}, t) = \sum_n \sum_{\mathbf{p}_n} a_n \phi_n(\mathbf{x}) u_n(t) \tag{23}$$

where $\sum_{\mathbf{p}_n}$ stands for the integral over on-shell momenta and a_n are the normalized coefficients of the Fourier expansion - see [32] for more details. Being on-shell fields, after decompactification we find that the spatial components satisfy the equations of motion

$$(\nabla^2 + k_n^2)\phi_n(x) = 0 , \tag{24}$$

and both momentum and energy eigenmodes are orthogonal and complete. Using Poisson summation,[11] the energy eigenmodes complete set is such that

$$\int_0^{\lambda_x} \frac{dx}{\lambda_x} \phi_n^*(x)\phi_m(x) = \delta_{n,m} \quad , \qquad \sum_n \frac{\phi_n^*(x)\phi_n(y)}{\lambda_x} = \sum_{n'} \delta(x - y + \lambda_x n') . \tag{25}$$

The second relation shows that the spatial points, similarly to the time points, are defined modulo λ_x translations. Even though the whole information is in a single elementary space period λ_x, we can always write the above conditions extending the integration over the whole spatial region V_x where the field is supposed to be free and with an integer number of periods. In this case the substitution is just $\int_0^{\lambda_x} dx/\lambda_x \to \int_{V_x} dx/V_x$, where $V_x = N\lambda_x$ and $N \in \mathbb{N}$ is the integer number of periods in V_x.

2.1. Schrödinger equation and Hilbert space

The time evolution for the energy eigenmodes of the relativistic periodic field eq.(23) is described by the "bulk" equations of motion

$$(\partial_t^2 + \omega_n^2)u_n(t) = 0 , \tag{26}$$

where the frequency spectrum is fixed by boundary conditions, eq.(3). It is given in eq.(12) and in eq.(18) for massless and massive periodic fields respectively. These equations of motion along the time can be interpreted, together with the de Broglie relation eq.(11), as the Schrödinger equation. Since the energy eigenmodes eq.(23) satisfy $i\partial_t \phi_n(x)u_n(t) = \omega_n \phi_n(x)u_n(t)$, we obtain indeed the Schrödinger equation for the field

$$i\hbar\partial_t \phi_n(x, t) = E_n \phi_n(x, t) , \tag{27}$$

which is the "square root" of the eq.(26), see also [16, 17]. Roughly speaking, this is due to the fact that the Klein-Gordon equation is indeed the relativistic generalization of the Schrödinger equation.

---

[11]The Poisson summation implies that $\sum_n e^{-in\alpha} = 2\pi \sum_{n'} \delta(\alpha + 2\pi n')$.

Another important point is that we are describing standing waves. Therefore this is the typical case where a Hilbert space can be defined. Because of the orthogonality and completeness relations in eqs.(25) between the energy eigenmodes, it is natural to introduce the following inner product

$$\langle \phi | \chi \rangle_H \equiv \int_0^{\lambda_x} \frac{dx}{\lambda_x} \phi^*(x) \chi(x) \ . \tag{28}$$

This naturally yields to an Hilbert space with the following eigenstates

$$\langle x | \phi_n \rangle_H \equiv \frac{\phi_n(x)}{\sqrt{\lambda_x}} \ . \tag{29}$$

Furthermore, we can build the Hamiltonian operator as

$$\hat{H} \, | \phi_n \rangle_H \equiv \hbar \omega_n \, | \phi_n \rangle_H \ . \tag{30}$$

From the eq.(26) the time evolution for a generic state $| \phi(0) \rangle_H = \sum_n \alpha_n \, | \phi_n \rangle_H$ can now be written as

$$| \phi(t) \rangle_H = \sum_n e^{-i\omega_n t} \alpha_n \, | \phi_n \rangle_H \ , \tag{31}$$

that is, using the Hamiltonian operator [9], we can equivalently write

$$| \phi(t) \rangle_H = e^{-\frac{i}{\hbar} \hat{H} t} \, | \phi(0) \rangle_H \ . \tag{32}$$

The Schrödinger equation can be written in a more familiar form

$$i\hbar \partial_t \, | \phi(t) \rangle_H = \hat{H} \, | \phi(t) \rangle_H \ . \tag{33}$$

We are assuming that the operator $\hat{H}$ is not a function of time (no source terms and no interactions in order to preserve periodicity inside the volume V_x). It corresponds formally to the generator of time translations

$$U(t', t) = e^{-\frac{i}{\hbar} \hat{H} (t - t')} \ . \tag{34}$$

This time evolution between generic t' and t'' can be justified by complex dynamics caused by the periodic time dimension and it has the Markovian operator property

$$U(t'', t) U(t, t') = U(t'', t') \ ; \qquad t'' \geq t \geq t' \ . \tag{35}$$

Using this property we divide the time interval in N elementary intervals of length ϵ

$$U(t'', t') = \prod_{m=0}^{N-1} U(t' + t_{m+1}, t' + t_m - \epsilon) \ ; \qquad N\epsilon = t'' - t' \ , \tag{36}$$

where we are using the notation $t_{m+1} = (m + 1)\epsilon$ and $t_m = m\epsilon$. Notice that for an Euclidean time, $U(t + \epsilon, t)$ is analogous to the transfer matrix of classic statistical mechanics, so a statistical interpretation of the periodic dynamics is possible.

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2.2. Feynman path integral

We point out that, without any further assumption than periodicity, all the ingredients to build a path integral are already contained in this periodic field theory: we have a Hamiltonian time evolution operator eq.(34), with the Markovian property eq.(36) and a complete set of energy eigenfunctions eqs.(25). From a mathematical point of view and proceeding completely in standard way, we make use of the completeness and orthogonality relations of the $\phi_n(x)$ in eq.(36). Separating the space-time evolution in infinitesimal parts we get

$$
\begin{aligned}
U(x'', t''; x', t') \;=\; & \int_0^{\lambda_x} \left(\prod_{m=1}^{N-1} \frac{dx_m}{\lambda_x} \right) U(x'', t''; x_{N-1}, t_{N-1}) U(x_{N-1}, t_{N-1}; x_{N-2}, t_{N-2}) \\
& \times \;\; \dots U(x_2, t_2; x_1, t_1) U(x_1, t_1; x', t') \;.
\end{aligned}
\tag{37}
$$

The elementary periodic evolutions between spatial points $x_k = x(t_k)$ to $x_{k-1} = x(t_{k-1})$ turn out to be given by

$$
U(x_{m+1}, t_{m+1}; x_m, t_m) \;=\; \sum_{n_m} e^{-\frac{i}{\hbar}(E_{n_m} \Delta \epsilon_m - p_{n_m} \Delta x_m)} \;,
\tag{38}
$$

with the notation $\Delta x_m = x_{m+1} - x_m$ and $\Delta \epsilon_m = t_{m+1} - t_m$. As already mentioned the energy spectrum is $E_n(\bar{p}) = n\hbar\bar{\omega}(\bar{p})$. These elementary space-time evolutions correspond to the "unitarized" periodic field $\phi(\Delta x_m, \Delta \epsilon_m)$ (that is to say a periodic field with unitarized coefficients $a_n \equiv 1, \; \forall n$). Using Dirac notation eq.(28) - see also the definition of the expectation value eq.(41) in the next section - we get the familiar form

$$
U(x_{m+1}, t_{m+1}; x_m, t_m) = \phi(\Delta x_m, \Delta \epsilon_m) \;=\; \lambda_x \langle \phi | \, e^{-\frac{i}{\hbar}(\hat{H} \Delta \epsilon_m - \hat{p} \Delta x_m)} \, | \phi \rangle_H \;,
\tag{39}
$$

where the operator $\hat{p}$ is defined in analogy with the Hamiltonian operator in eq.(30). Plugging these microscopic evolutions in eq.(37) we get formally the Feynman path integral in phase space for a time independent Hamiltonian

$$
U(x'', t''; x', t') = \lim_{N \to \infty} \int_0^{\lambda_x} \left(\prod_{m=1}^{N-1} dx_m \right) \left\{ \prod_{m=0}^{N-1} \left[\langle \phi | \, e^{-\frac{i}{\hbar}(\hat{H} \Delta \epsilon_m - \hat{p} \Delta x_m)} \, | \phi \rangle_H \right] \right\} \;.
\tag{40}
$$

Remarkably this fundamental result has been obtained just assuming relativistic periodic waves without any further assumption such as commutation relations between the canonical variables. We will see in sec.(2.3) that commutation relations can be derived directly from periodic fields, but it can be obtained from this path integral as well. Assuming that in the non-relativistic limit only the first mode ($n = 1$) is largely populated it is possible derive the non-relativistic free particle limit of the theory [32].

The path integral formulation arises as a direct consequence of the fact that the non trivial periodic dynamics yield a class of equivalence between initial and final points translated by space-time periods. It is possible to reach a given final configuration following a class of periodic on-shell paths, *i.e.* paths with different winding numbers. In other

words, contrarily to the Feynman formulation where there is a unique classical path, a periodic field interferes with itself because of the periodic boundary conditions and the consequent equivalence class of paths with different winding numbers, without relaxing the validity of the least action principle.

2.3. Commutation relations

In sec.(2.1) we found that periodic fields can be written in a Hilbert space with time evolution given by the Schrödinger equation. Now we would like to have commutation relations in order to formalize the analogy with the canonical formulation of quantum mechanics as well. Looking at the inner product in eq.(28) we identify the mean value of a given observable $F(x)$ between generic initial and final states $|\phi\rangle$ and $|\chi\rangle$ as

$$\langle\chi(x_f,t_f)|F(x)|\phi(x_i,t_i)\rangle_H =$$
$$= \int_0^{\lambda_x} \frac{dx}{\lambda_x} \sum_{n,m} \alpha^*_{\chi m} e^{i\omega_{\chi m}t_f - ik_{\chi m}(x_f-x)} F(x) e^{-i\omega_{\phi n}t_i + ik_{\phi n}(x-x_i)} \alpha_{\phi n} \ , \qquad (41)$$

where λ_x is the spacial period in x - the integration volume can be extended to the whole periodic region V_x. To determine commutation relations we follow [36], but using directly the unitarized periodic fields $\phi(x,t)$ rather than the periodic path integral. In fact [32] there is an equivalence between the two formulations. We continue by evaluating the mean value of $\partial_x F(x)$. Integrating by parts eq.(41) and considering the periodicity λ_x of the spatial variable and of the states,[12] we get

$$\langle\chi(x_f,t_f)|\partial_x F(x)|\phi(x_i,t_i)\rangle_H =$$
$$= \frac{i}{\hbar} \int_0^{\lambda_x} \frac{dx}{\lambda_x} \sum_{n,m} \alpha^*_{\chi m} \alpha_{\phi n} e^{i\omega_{\chi m}t_f - ik_{\chi m}(x_f-x)} [p_{\chi m} F(x) - F(x) p_{\phi n}] e^{-i\omega_{\phi n}t_i + ik_{\phi n}(x-x_i)}$$
$$= \frac{i}{\hbar} \langle\chi(x_f,t_f)|\hat{p}F(x) - F(x)\hat{p}|\phi(x_i,t_i)\rangle_H \ . \qquad (42)$$

Thus, by choosing $F(x) \equiv x$, it turns out that

$$\langle\chi(x_f,t_f)|1|\phi(x_i,t_i)\rangle_H = \frac{i}{\hbar} \langle\chi(x_f,t_f)|\hat{p}x - x\hat{p}|\phi(x_i,t_i)\rangle_H \ , \qquad (43)$$

which, for generic initial and final states, reproduces the commutation relation of quantum mechanics

$$[x,\hat{p}] = i\hbar \ . \qquad (44)$$

We thus find that, besides the Feynman formalism derived in the previous subsection, the fundamental elements to build the canonical formulation of quantum mechanics are already contained in this theory as well.[13]

---

[12]It is equivalently possible to assume Dirichlet boundary conditions at the boundaries $\Phi(0,t) = \Phi(\lambda_x,t) \equiv 0$ instead of periodicity. In this way the only inessential difference is that there is no zero mode $n = 0$, that is no translational mode.

[13]More easily we note that $[x,-i\hbar\partial_x]\Phi(x,t) = i\hbar\Phi(x,t)$.

2.4. Heisenberg uncertainty relation

For a periodic wave it is possible to obtain an uncertainty rule in a rather immediate and trivial way. To determine the frequency of a free wave and thus the energy of the related mode we must count the oscillations for at least a time interval greater than the fundamental period: the longer the measuring time, the lower the frequency uncertainty. Mathematically we can see this by noting that the phase $\bar{E}t/\hbar$ is defined modulo factors $2\pi n$. Supposing for simplicity $n = 1$, we can reabsorb this factor either as a variation of the time variable $\Delta t = 2\pi\hbar/\bar{E}$ or of the energy $\Delta E = 2\pi\hbar/t$, so that $\Delta E \times \Delta t = (2\pi\hbar)^2/\bar{E}t$, which is minimized by the largest value of the time in the denominator $t \to T_t$. Finally, we recover the Heisenberg uncertainty relation[14]

$$\Delta E \times \Delta t \geq 2\pi\hbar = h \ . \tag{45}$$

This is a direct consequence of the de Broglie assumption in eq.(11), that can be generalized [9] to

$$E_n R_t = n\hbar \ . \tag{46}$$

This relation can be regarded as the semi-classical Einstein's formulation of the Bohr-Sommerfeld quantization condition: in a given potential only phase-space orbits which fit in an integer number of periods T_t are allowed. This simple recipe is sufficient to solve many problems of non-relativistic quantum mechanics[15], such as the quantum harmonic oscillator, the anharmonic or anisotropic quantum oscillator, linear potential, of the various well potentials and Dirac delta potentials, the hydrogen atom, etc... [32].

3. Quantum mechanical interpretation

Since we have inferred the Hilbert space eqs.(25), eq.(28) and eq.(29), the Schrödinger equation eq.(27), eq.(30) and eq.(33), the commutation relations eq.(44), the path integral eq.(40), the Heisenberg uncertainty relation eq.(45) and the Bohr-Sommerfeld condition eq.(46) from the periodicity assumption, it is reasonably correct to interpret our theory as a quantum theory.

In general, the standard fields can be thought of as an integral over elementary harmonic oscillators with angular frequencies $\bar{\omega}(\bar{\mathbf{p}})$. In the usual formulation a non interacting classical field is a single de Broglie plane wave with fixed momentum $\bar{\mathbf{p}}$ and therefore can be described by a single harmonic oscillator with characteristic periodicity $T_t(\bar{\mathbf{p}}) = 1/\bar{\omega}(\bar{\mathbf{p}})$. Its angular frequency $\bar{\omega}(\bar{\mathbf{p}})$ must be written as in eq.(12) or eq.(18) respectively for massless or massive fields.

The usual quantization of bosonic fields (namely the second quantization) is obtained by explicitly quantizing each harmonic oscillator, that is by imposing commutation relations After normal ordering, every single harmonic oscillator has a quantized energy

---

[14]Taking into account the square modulo of the wave function we have a phase invariance $n\pi$ which gives a factor $1/2$ in the final result.

[15]Modulo the zero-point energy which must be fine-tuned using twisted boundary conditions.

spectrum $:E_n(\bar{\mathbf{p}}): = :\hbar\omega_n(\bar{\mathbf{p}}): = n\hbar\bar{\omega}(\bar{\mathbf{p}})$.[16] These are just the admitted energies of a periodic field with periodicity T_t as prescribed by eq.(11). On the other hand, all the arguments given so far can easily be generalized to the orbifold case $t \in \mathbb{S}^1/\mathcal{Z}_2$, which gives the spectrum with vacuum energy $E_n(\bar{\mathbf{p}}) = (n + 1/2)\,\hbar\bar{\omega}(\bar{\mathbf{p}})$, by supposing that the field is odd under the $\mathcal{Z}_2$ parity (antiperiodicity). Because of the similarities with finite temperature field field theory, for the scope of this paper fermions can be thought of as antisymmetric fields. Similarly, a generic value of the vacuum energy $v\hbar\bar{\omega}$ can be obtained by assuming twisted periodic boundary conditions $\Phi(\mathbf{x},t) = \exp\left[-2\pi v\right]\Phi(\mathbf{x}, t + 2\pi R_t)$. Anyway, as explained in more detail in the sec.(3.1) and [40, 41, 32], these contributions to the energy are of "little importance" since they come from phase factors in front of the field. Furthermore, we point out that "the Casimir effect, often invoked as decisive evidence that the zero point energies of the quantum field are real [...] can be formulated and the Casimir forces can be computed without reference to zero point energies", but assuming a generalization of the van der Waals force [42] in the interaction between the electrons in the two metallic plates. Because of the analogy with finite temperature field theory and in spite of the fact that we are dealing with bosonic fields, it is natural to assume the fermionic fields to be antiperiodic in order to satisfy the spin-statistics relation [5]. Indeed, under this hypothesis fermionic fields have vacuum energy $\hbar\bar{\omega}(\bar{\mathbf{p}})/2$.

We summarize the analogy between periodic fields and quantum fields that we want to explore by saying that every relativistic field $\Phi(\mathbf{x}, t)$ with assigned four-momentum $\bar{p}_\mu$ has a fixed space-time periodicities $T_\mu = h/\bar{p}^\mu$, in agreement with the de Broglie hypothesis. It can be decomposed in a series of eigenstates $\phi_n(\mathbf{p})$ with energies $n\hbar\bar{\omega}(\bar{\mathbf{p}})$ whose interpretation is in terms of the "quanta" of the related quantum field. Further evidences for this mapping with ordinary quantum mechanics are given in [32] where we describe the essential phenomenology and elucidative applications.

3.1.   Determinism

Another important aspect which motivated the investigation upon periodic time dimension is the 't Hooft determinism [23, 27, 22] where there is a "close relationship between the quantum harmonic oscillator and a classical particle moving along a circle". We approach the 't Hooft determinism by assuming periodic fields with time period T_t on a lattice with N sites, in order to (de)construct [25, 26] the time dimension.[17] We associate to every discretized phase, $i.e$ to every site of the lattice, a column state $|0\rangle, |1\rangle, \ldots, |N - 1\rangle$. The model is analogous to an harmonic system of N masses and springs on a ring. It turns out that if the time accuracy is $\Delta t \gg T_t$, at every observation the field $\Phi(x, t)$

---

[16]According to the Born rule, we assume that the probability density $\rho = |\Phi(\mathbf{x}, t)|^2$ associated to the periodic fields is given by the inner product $\langle\Phi(\mathbf{x}, t)|\Phi(\mathbf{x}, t)\rangle_H$, eq.(28). ρ corresponds to the non-relativistic limit of the charge density j_0 related to the periodic field $\Phi(\mathbf{x}, t)$ (for instance we may also note that when we observe a particle we inevitably stop it on the rest frame of the detector [37, 38, 39]) - see for more details [32].

[17]In [26] a four-dimensional Yang-Mills field theory emerges dynamically by dimensional (de)construction mechanism [25, 26] applied to replicated three-dimensional gauge theories (a moose model in three-dimensions). This dynamically constructed periodic time dimension leads to the Heisenberg uncertainty relation eq.(45) and energy quantization eq.(8).

appears in an arbitrary phase $|n\rangle$ of its cyclic evolution, so that the evolution has an apparent aleatoric behavior; as if observing a clock under a stroboscopic light [27], or a dice rolling to fast to predict the result. In fact, as already discussed in sec.(1.2), if the underlying periodic dynamics are too fast to be observed ($\lesssim 10^{-20}s$), the time evolution between two column states $|n\rangle$ can only be described statistically through the operator $\tilde{U}(\Delta t = \epsilon) = \exp[-\frac{i}{\hbar}\tilde{H}\epsilon]$, where $\tilde{H}$ is a $N \times N$ matrix. In the limit of large N the column states obey to the relation $\tilde{H}|n\rangle \sim \hbar\bar{\omega}(n + 1/2)|n\rangle$. This reproduces just the energy eigenvalues of the quantum harmonic oscillator, apart for a phase factor of "little importance" in front of the operator $\tilde{U}(\epsilon)$ which reproduces the factor $1/2$ in the eigenvalues [22, 24, 40, 41]. From the evolution operator $\tilde{U}(t)$ we can once again observe the analogy between quantum and statistical mechanics.[18] Due to the extremely fast underlying dynamics we loose information about the fundamental classical theory which give rise to the quantum behavior. For this reason we can speak about deterministic or pre-quantum theories.[19] Finally, since a periodic time dimension can induce periodicity to the proper time, that is the worldline parameter of the our fields, we have an analogy to string theory where one of the two worldsheet parameters is compactified.

Motivated by the 't Hooft determinism and the attempts to quantize gravity, a model of a classical particle moving in five-dimensions, two of which are compactified on a torus, is illustrated in [27, 28, 29, 30, 46]. The ergodic dynamics associated to this model give rise to an effective time and thus to a so called "stroboscopic quantization". The relevant idea here is the derivation of a notion of "time" which emerges from the "ticks" of an ergodic system. Similarly, in our theory the notion of time emerges from the "ticks" provided by the de Broglie internal clocks. Geometric quantization [47, 48] seems to indicate another connection between the notion of time and quantization. In fact, in this theory some quantum phenomena emerge by integrating out two grassmannian partners of the physical time.

In the 't Hooft approach to determinism as well as in the model with "stroboscopic quantization" there is the attempt to avoid local hidden variables. It is worth noting that the approach with compact time proposed throughout this paper has not local-hidden-variables that must be integrated out to get the quantum observables, we have just space and time coordinates which are physical variables. On the other hand the periodic conditions in eq.(3) can be regarded as an element of non locality (which is consistent with relativistic causality) in the theory. Therefore model proposed in this paper is deterministic[20] and it represents a possible way out of the Bell's inequality or similar non-local-hidden-variable theorems [31].

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[18]Here we note also a resemblance with the random walk problem which was originally solved using its analogy with interference of iso-periodic waves with random phase distribution [43].

[19]Further similitudes with the 't Hooft determinism are given by the fact that the de Broglie time periodicities can be regarded as "cellular automata" [44] and our fields, being constrained in a periodic time dimension, share interesting analogies with black-hole thermodynamics [45].

[20]Here we mean mathematical determinism. From a practical point of view is in fact impossible to measure time with an infinite accuracy and thus to know the exact boundary conditions of the system under investigation, see sec.(2.4).

3.2. Compact space-time formalism

Paraphrasing the Newton's law of inertia and the de Broglie hypothesis of periodicity, we assume that every isolated elementary system (every free elementary field) has persistent and constant time periodicity (as long as it doesn't interact) fixed by the inverse of the energy $T_t = h/\bar{E}$. Considering the periodicities induced on the modulo of the spatial dimensions, the resulting space-time periodicities are those of the ordinary de Broglie waves and therefore they are consistent with special relativity.

A conceptual effort is required for a conceptual understanding of this theory, because it adds a property of periodicity to our ordinary notion of relativistic time. From a formal point of view, in this relativistic theory the physical time is well defined through the relation between periodicity and energy. It respects all the required properties such as Lorentz transformations, causality and chronological ordering. But, as much as the Newton's law of inertia doesn't imply that every point particle goes in a straight line, our assumption of periodicity does not mean that the physical world should appear to be periodic. In fact there is not a single static periodicity which would serve as privileged reference. On the contrary elementary systems (that we represent as fields) at different energies have different periodicities.[21] The conjecture is that the combination of these different periodicities, that for massless fields may effectively vary between the Planck time $\sim 10^{-44}s$ to the age of the Universe $\sim 10^{15}s$ or more (in the hypothesis of a cyclic universe), is the reason of our perception of the time flow. Furthermore, through interactions the elementary systems pass from a periodic regime to another periodic regime, forming in general ergodic and even more chaotic evolutions, giving rise to a possible statistical interpretation of the arrow of time.

To figure out the possibility of a formulation of relativistic fields in compact space-time dimensions we follow few simple logical steps. Ordinary field theory is based upon de Broglie waves that are then quantized by imposing commutation relations. To every de Broglie wave there is associated a frequency proportional to its energy and thus an intrinsic periodicity which is usually called de Broglie internal clock. In fact time can be only defined by assuming periodicity, in order to ensure that the duration of a unit of time is always the same; in past, in the present and in the future. Our usual - non compact - time axis is defined with reference to the Cs-133 atomic clock whose period is about $10^{-10}s$, an electron at rest has an internal de Broglie clock of about $\sim 10^{-20}s$ whereas an hypothetical heavy particle of 1 TeV has an internal clock of $\sim 10^{-27}s$. Depending on its energy, a massless field such as the electromagnetic field (or the gravitational field), can in principle have all the possible intrisic periodicities. In particular it can have an infinite period (or of the order of the age of the universe).

Every value of our time axis is characterized by a unique combination of phases of all the de Broglie clocks of the elementary fields constituting the system under investigation. This means that the external time axis can be dropped and the flow of time can be effectively described using the de Broglie internal clocks as in a calendar or in

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[21]This concept has a precise mathematical justification, in fact Fourier showed that every regular (not necessarily periodic) function can be expressed as an integral over periodic functions.

an stopwatch - the massless fields provide the long time scales. This is a simplified picture since we must remember that the clocks can vary periodicity through interaction (exchange of energy), that periods depend on the reference systems according to the relativistic laws and that the combination of two clocks with irrational ratio of periodicities gives ergodic (not exactly periodic) evolution. It is interesting to note that this picture is of particular interest for the problem of the time symmetry in physics, in fact the de Broglie clocks can be supposed to be equivalently supposed to be clockwise or anticlockwise. Remembering the Einstein's definition of relativistic clock [1] (see introduction), we can restrict our attention on a single period of every de Broglie internal clock, that is of every elementary field constituting our system. This means the physical information of the fields is contained in the single periods, therefore we formalize this by investigating fields in compact time and periodic (or Dirichlet) boundary conditions. Similar argumentations hold for the spatial dimensions. In the non relativistic limit, matter fields can be approximated as with infinite spatial periodicity and microscopic time compactification proportional to its Compton wavelengths. Hence they can be regarded as nearly three spatial dimensional objects. Furthermore, since they are spatially localized inside their microscopical Compton wavelengths, they can be effectively regarded as non-relativistic point-like particles.

Another intuitive image can be found in the many similarities with acoustic waves [49]. The sound is a set of standing waves generated by a string, a membrane or a solid body vibrating in one, two or three compact spatial dimensions respectively. The harmonics (frequency eigenstates) of these acoustic waves are those allowed by the size of the spatial compact dimensions in which the sound source is embedded. In a full relativistic generalization of the sound waves, our relativistic fields can be thought of as being generated by vibrating objects (sources) characterized by intrinsically compact space-time dimensions. Roughly speaking, massless fields at small momentum have nearly infinite time periodicity (nearly continuous energy spectrum) so that they act like the medium whereas matter fields at small momentum have compact time dimension and they act like sources. The difference with the usual field picture is that now we allow a "timbre" to the de Broglie waves, that is we consider all the frequencies, and thus the different spectral composition, allowed by the space-time periodicities $2\pi R_\mu$.

3.3. Towards the formalization of interactions

So far we have illustrated the formal and conceptual correspondences between a field theory with periodic time dimension and the usual quantum theory, concerning only with free field. The exact solution of the interaction between periodic fields and thus the transition between to different periodic regimes is beyond the scope of this article. Most likely, it would require the development of a perturbative theory, adding an interaction term to the Hamiltonian of the periodic path integral eq.(40).

To give a qualitative picture of the interacting periodic fields, the most trivial example is Compton scattering $e' + \gamma' \rightarrow e'' + \gamma''$. As already mentioned about fig.(3), we must merely consider the energy-momentum conservation in terms of conservation of the in-

verse of the space-time periodicities of the fields involved, $1/T_\mu^{\gamma'} + 1/T_\mu^{e'} = 1/T_\mu^{\gamma''} + 1/T_\mu^{e''}$. The change in periodicity of a field during the interaction (that is when the field has significative overlaps or interference) can be regarded as a deformation of the space-time compactification lengths. This problem can be equivalently reformulated by keeping the compactification lengths constant and and assuming a deformation of the metric. Hence interaction can be interpreted in terms of geometrodynamics. This argumentation leads to field theories on curved space-time. For instance [32], as we will expose in a dedicated paper, we can imagine to prepare a volume of quarks and gluons at high energy, for instance using a collider. The system looses energy by radiating hadronically or electromagnetically [50]. In first approximation, as predicted by the hydrodynamic Bjorken model [51] and in similitude with thermodynamic system [50], the quark-gluon plasma passes exponentially from a high energy regime characterized by small periodicities, to a low energy regime characterized by large periodicities. This conformal exponential dilatation of the space-time periodicities turns out to be described by five-dimensional fields with zero five-dimensional masses embedded in a (virtual) AdS metric. From the mapping with quantum mechanics described so far, by imposing such a dilatation of the periodicities we expect to observe an evolution of the quantum observables with the energy. In particular this implies that the gauge coupling has a logarithmic running with the energy [52, 53]. In fact, it is well know that the classical correlator of a classical field in a warped background can be approximatively matched with the quantum two point function of QCD. Hence, we get a close parallelism with the AdS/CFT correspondence which originally motivated our study. I fact, interpreting the Maldacena conjecture [54] as in Witten's work [55], it describes a parallelism between classical fields in a warped dimension and quantum phenomena in a lower dimensional theory, that is it encodes the quantum behavior in classical configurations of an (warped) extra dimension.

Conclusions

We investigated the hypothesis of dynamical and local space-time periodicities, extending the "old quantum theory". Since these periodicities are the natural de Broglie periodicities of the classical fields, the resulting theory respects Lorentz invariance, preserves causality, allows time ordering, and reproduces the relativistic field theory in the limit of no boundaries. In fact, periodic conditions imposed to the relativistic waves, similarly to the usual boundary conditions, minimize the relativistic action. Indeed, special relativity prescribes that time is a local and dynamical property. In fields with compact space-time dimensions this property is manifest through the inverse proportionality between the energy and the time periodicity. We found that a massive periodic field, whose characteristic rest spatial width is its Compton length, has a extremely fast intrinsic periodicity ($\lesssim 10^{-20} s$) fixed by the inverse of its mass, as conjectured by de Broglie. The space-time periodicities are different if observed from different inertial frames, in agreement with Lorentz transformations and relativistic dispersion relations. Hence, the theory is covariant.

The study of the compactification of the time dimension has highlighted remarkable

connections between relativistic, quantum and thermal theories. We pointed out several correspondences to the usual quantum theory such as the arising of a discretized energy spectrum, of commutation relations and of uncertainty relations. The effective time evolution of a periodic field is described by the Schrödinger equation in a Hilbert space. Due to invariance by space-time translations of periods there are different classical trajectories with different winding number between the initial and final points. This gives rise to interference between different on-shell paths and thus to a path integral formulation, without relaxing the variational principle. As a consequence of the periodic nature of the fields, typical quantum phenomena such as black body radiation, the double slit experiment, Schrödinger problems, superconductivity, and many others can easily be reformulated. The connection with thermal theory comes because of the close analogy with the finite temperature field theory, and because of the underlying statistical laws. Indeed we have tried to construct a consistent description of these three theories using the simplest physical system possible, essentially waves with boundary conditions.

The field theory proposed here is a good candidate for pre-quantization since quantization arises from a deterministic theory instead of being imposed. As the AdS/CFT correspondence, which seems to have an immediate interpretation in this theory, the results obtained so far are non trivial. They seem to open a new scenario where a compact time dimension arises as something more physical than a simple mathematical trick, as believed in finite temperature field theory. Indeed, a dynamically compact Minkowskian time leads to the concrete possibility to combine special relativistic and quantum theory in a deterministic wave theory. The great advantages of such deterministic theory can be potentially extended to all quantum mechanical applications but especially in those branches of physics where the quantum and relativistic mechanics are difficult to conciliate, such as some aspects of high energy quantum field theory and quantum gravity.

The concept of time arising from this theory satisfies all the requirements prescribed by special relativity and, combining the different de Broglie internal clocks of the elementary fields as in a calendar or in a clock, we can indeed fix and order events in time. The non periodic phenomena that we observe can be easily explained by the fact that systems can pass from a periodic regime to another through interactions (energy exchange). If non periodic systems or similarly systems with periodicities larger than our observation time are interacting with the elementary system we are measuring, its periodic evolution will be no more manifest.[22]

Time has been defined by counting the number of oscillations of the Cesium atom or of the incense lamp of the Pisa Dome, the number of the orbits of the Earth or of the Moon. But these definitions inevitably make use of the *a priori* assumption of periodicity of isolated elementary systems and, "by the principle of sufficient reason", we assume that the whole information of these elementary systems is encoded in a single period, as implicitly said by Einstein himself [1] in his definition of a relativistic clock. For this reason, and for the ones mentioned in this work, we consider it worth investigating the physical consequences of an intrinsically cyclic nature of time.

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[22]For instance the universe can be cyclic or not, and with respect to this master time scale more and more events appear to have or have not a periodic nature.

Acknowledgements

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