Geometric Aspects of the Standard Model and the Mysteries of Matter

Florian Scheck
Institute of Physics, Theoretical Particle Physics
Johannes Gutenberg–University
D-55099 Mainz (Germany)
(December 23, 2009)

Contents

Geometric and Topological Methods for Quantum Field Theory
Summer School, Villa de Leyva, July 6 – 23, 2009

1 Radiation and matter in gauge theories and general relativity

The basic structure of gauge theories seems to distinguish *radiation* from *matter* as two categories of different origin. The massive and massless vector or tensor bosons, the photon, the $W^{\pm}$- and Z^0-bosons, the gluons, and the graviton, respectively, which are the carriers of the fundamental forces, belong to what may be termed *radiation*. Here we allude to the analogy to (quantum) electrodynamics described by Maxwell's equations and to Einstein's equations for General Relativity (GR). They are described by geometric theories, i.e. Yang-Mills (YM) theories or, in the case of GR, by semi-Riemannian geometry in dimension four. To a large extent, they are *classical* theories.
Matter, i.e. quarks and leptons and composites thereof, a priori, seems to belong to a different kind of physics which, at first sight, does not exhibit an underlying geometrical structure. While the equations of motion of YM theories and of GR, taken in isolation, describe nontrivial physics, matter cannot "live on its own" without the gauge bosons that mediate the fundamental interactions – unless one is satisfied with a theory of free particles which remains untestable in experiment. As soon as one enters the quantum world, however, the two categories, like two rivers, start mingling their waters:

(A) The Higgs particle, as a most prominent example of current interest in particle physics, plays a rather enigmatic role. Its phenomenological role in providing mass terms for some of the vector bosons and for the fermions of the theory suggests that it be another form of "matter", beyond ordinary matter made up of quarks and leptons. Models based on noncommutative geometry, in turn, classify the Higgs field in the generalized YM connection, besides the gauge bosons, and hence declare it to be part of "radiation". As such, it generates parallel transport between universes which are separated by a discrete distance.

(B) The requirement of *renormalizability* of quantum gauge theories with massive vector bosons entails the introduction of so-called Stückelberg (scalar) fields whose place in the classification needs to be clarified.

(C) Quarks and leptons are described by a *Dirac operator* which in its mass sector exhibits a significant, though mysterious structure. Dirac operators, in turn, are the driving vehicles in constructing noncommutative geometries designed to generalize YM theory – and to describe the standard model! They act on the Hilbert space which is spanned by the myriad of quark and lepton states.

(D) Quantization of YM theories is possible only in the absence of *anomalies*. These, in turn, depend on the classification of the matter particles with respect to the structure group. In the minimal electroweak standard model, for instance, it needs a conspiracy between the three generations of quarks and of leptons to render it renormalizable.

(E) As emphasized particularly by Scharf and his group, the requirement of BRST (Becchi, Rouet, Stora, Tyutin) invariance of the underlying Lagrangian fixes much of its structure and, thereby, intertwines the radiation sector and the matter sector.

(F) Last not least, the currents and charges of matter or its energy-momentum tensor act as the sources in the equations of motion of YM and gravitational fields, respectively.

In these lectures we work out several of the themes alluded to above, both by way of construction and by means of instructive examples. We start with a schematic description of YM theories including spontaneous symmetry breaking (SSB) within the classical geometric framework, and including matter particles. In a first excursion to quantum field theory we describe the stratification of the space of connections and its relevance to anomalies. In order to clarify the phenomenological basis on which YM theories of fundamental interactions are built, we describe some of the most pertinent phenomenological features of leptons and of quarks. Via the Dirac operator describing leptons and quarks we turn to constructions of the standard model in the framework of noncommutative geometry. This, in turn leads us to a closer analysis of the mass sector and state mixing phenomena of fermions. The intricacies of quantization are illustrated by a semi-realistic model for massive and massless vector bosons.

1.1 Schematic construction of gauge theories

The backbone of a gauge model is a *structure group* G which is taken to be simple or semi-simple. For physical reasons *compactness* of G is essential, but why? (Exercise 1). For instance, the electroweak sector of the minimal standard model (SM) is based on

$$G_{\text{ew}} = \mathrm{U}(2) = \left\{ \mathbf{U} \in M_2(\mathbb{C}) | \, \mathbf{U}^\dagger \mathbf{U} = \mathbb{1} \right\} \ ,$$

which splits into the SU(2) of what we call weak isospin, and the U(1) of weak hypercharge. As is well-known, the full SM is built on the structure group

$$G = \mathrm{U}(2) \times \mathrm{SU}(3) \ ,$$

with the SU(3) of color interactions included. The model is formulated on a principal fibre bundle

$$\mathcal{P} = \left(P \stackrel{\pi}{\longrightarrow} M, G \right) \ ,$$

where, barring relativity for the moment, $M = \mathbb{R}^{(1,3)}$ is flat four-dimensional Minkowski space. The structure group G then is replaced by the *gauge group* $\mathfrak{G}$. In geometrical terms, this is the group of automorphisms of the principal bundle $\mathcal{P}$ which commutes with the right action R_g of G and which maps every fibre onto itself, viz.

$$\Psi \in \mathfrak{G} \ , \quad \Psi : \mathcal{P} \to \mathcal{P} \ , \tag{1a}$$

$$\pi\left(\Psi(z)\right) = \pi(z) \ , \quad \Psi(z \cdot g) = \Psi(z) \cdot g \ . \tag{1b}$$

As Ψ acts on fibres only, one has

$$\Psi(z) = z\gamma(z) \ , \quad z \in \mathcal{P} \ , \ \Psi \in \mathfrak{G} \ , \tag{2a}$$

$$\gamma : P \to G : z \mapsto \gamma(z) \ , \quad \gamma(z \cdot g) = \mathrm{Ad}\ g^{-1}\left(\gamma(z)\right) = g^{-1}\gamma(z)g \ . \tag{2b}$$

Thus, the gauge group $\mathfrak{G}$ can be identified with the set of maps $\gamma : \mathcal{P} \to G$, so that $(\gamma\gamma')(z) = \gamma(z)\gamma'(z)$.

The connection form $A \in \Omega^1(\mathcal{P}, \mathfrak{g})$ takes values in the Lie algebra Lie $G = \mathfrak{g}$. Its relation to what is called *gauge potential* in physics is effected by local sections

$$\sigma_i : U_i \subset M \to \mathcal{P},$$

such that

$$A^{(\sigma_i)} = \sigma_i^* \omega \ , \quad A^{(\sigma_i)} \in \Omega^1(U_i, \mathfrak{g}) \ , \ U_i \subset M \ .$$

Suppose $\{U_i\}$ is a covering of space-time M, and $\{(\varphi_i, U_i)\}$ are the charts of an atlas describing M. The connection as a whole is then obtained in the usual manner by joining the chart representations over a complete atlas.

1.1.1 Radiation

On Minkowski space which is flat and simply connected, the construction of the connection form reduces to[1] the definition

$$A := \mathrm{i}q \sum_{k=1}^{N} A^{(k)} \mathbf{T}_k \ , \quad N = \dim \mathfrak{g} \ , \tag{3a}$$

where the operators $\mathbf{T}_k$ are the generators of G, q is a generalized "charge", and $A^{(k)}$ are one-forms on $M = \mathbb{R}^{(1,3)}$,

$$A^{(k)} = A_\mu^{(k)}(x)\ \mathrm{d}x^\mu \ , \quad k = 1, 2, \ldots, N \ . \tag{3b}$$

The functions $A_\mu^{(k)}(x)$ are components of the gauge fields, linear combinations of which will describe the massive or massless vector fields of the SM. For example, if $A_\mu^{(1)}(x)$ and $A_\mu^{(2)}(x)$ denote the coefficients of the generators $\mathbf{T}_1$ and $\mathbf{T}_2$ of SU(2), the physical W-fields that mediate weak charged current interactions are given by

$$W_\mu^{(\pm)}(x) = \tfrac{1}{\sqrt{2}} \left(A_\mu^{(1)}(x) \pm \mathrm{i} A_\mu^{(2)}(x) \right) \ .$$

In Exercise 2 one is invited to verify that A is indeed the vehicle which is needed to perform *parallel transport* on the principal bundle.

From here on the construction is standard: Local gauge transformations on A are

$$A \longmapsto A' = gAg^{-1} + g\mathrm{d}(g^{-1}) \ , \quad g \in \mathfrak{G} \ . \tag{4}$$

[1]The Lie algebra valued one-forms $A^{(k)}$ are real forms. The optional factor i in the definition (3a) renders the operator A hermitean. This is useful in view of the hermiticity of Lagrangians or actions in physics.

The two pieces are seen to be a transformation by "conjugation", well-known e.g. from quantum mechanics, and a genuine local gauge transformation familiar from a U(1) theory such as Maxwell theory. Formally speaking, (4) is inhomogeneous and looks like a (generalized) affine transformation. The one-form A serves to construct the covariant derivative (cf. Exercise 2)

$$D_A = \mathrm{d} + A \, , \tag{5}$$

which reads, when applied to some (multiplet of) scalar field(s),

$$\partial_\mu \Phi(x) \to \left\{ \mathbb{1}\partial_\mu + \mathrm{i}q \sum_{k=1}^{N} A_\mu^{(k)}(x) \mathbf{T}_k \right\} \Phi(x) \, .$$

With respect to gauge transformations the covariant derivative transforms by conjugation only, $D_{A'} = g D_A g^{-1}$, there is no inhomogeneous term like for A. This is important for physics: A term such as $(D_A\Phi, D_A\Phi)$ where the brackets denote a G-invariant scalar product, is automatically invariant under *local* gauge transformations as well. In other terms, the brackets ensure *global* gauge invariance with respect to the structure group G, the covariant derivatives guarantee *local* gauge invariance with respect to the gauge group $\mathfrak{G}$.
The curvature two-form pertaining to the connection A is given by

$$F := D_A^2 = (\mathrm{d}A) + A \wedge A \, . \tag{6}$$

In contrast to D_A itself the action of D_A^2 ist *linear*, (Exercise 3). The analoga of the field strength tensor of Maxwell theory (familiar to physicists) are unveiled in the decomposition

$$F := \mathrm{i}q \sum_{k=1}^{N} \mathbf{T}_k \sum_{\mu<\nu} F_{\mu\nu}^{(k)}(x) \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \tag{7a}$$

in terms of ordinary, antisymmetric tensor fields $F_{\mu\nu}^{(k)}(x)$ and the base two-forms $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ on Minkowski space. If these tensor fieds are decomposed in terms of the component fields $A_\mu^{(k)}(x)$ a new feature appears as compared to Maxwell: The field strenghts are no longer linear in the A_μ fields,

$$F_{\mu\nu}^{(k)}(x) = \partial_\mu A_\nu^{(k)}(x) - \partial_\nu A_\mu^{(k)}(x) - q \sum_{m,n=1}^{N} C_{kmn} A_\mu^{(m)}(x) A_\nu^{(n)}(x) \, . \tag{7b}$$

Obviously, under a local gauge transformation F, Eq. (6), transforms by conjugation, $F' = gFg^{-1}$. The quadratic terms in (7b) which contain the structure constants C_{kmn} of the structure group, upon squaring F, lead to cubic and quartic interactions among the gauge fields. Indeed, the YM Lagrangian which must be a local invariant with respect to $\mathfrak{G}$, has the form

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4q^2 \kappa^{(\mathrm{ad})}} \, \mathrm{tr} \, \left(F_{\mu\nu} F^{\mu\nu} \right) \, , \tag{8}$$

where $\kappa^{(\mathrm{ad})}$ is the normalization of the trace

$$\mathrm{tr}\ (\mathbf{T}_i \mathbf{T}_j) = \kappa\, \delta_{ij} \tag{9}$$

in the adjoint representation (Exercise 4).
The result (8) shows that a world containing vector bosons only contains nontrivial physics. Suppose, for example, that the Lagrangian (8) describes the photon γ, the Z^0-boson and the charged $W^{\pm}$-bosons. (Of course, a mechanism is needed, in addition, which renders some of these massive and thereby defines the specific linear combinations of neutral fields which describe the photon and the Z^0.) The Lagrangian (8) produces well-defined coupling terms between these particles. For instance, the $WW\gamma$-vertex fixes the anomalous magnetic moment of the W^+ and W^- which, at least in principle, can be measured in scattering experiments. Other, perhaps more prominent and better known examples are provided by the three- and four-gluon vertices of quantum chromodynamics (QCD) which are tested in jet dynamics at e^+e^--colliders.

1.1.2 Matter

While the principle of local gauge invariance based on a group G fixes the structure of the *vector boson sector* to a large extent, the introduction of scalar or fermionic *matter fields* leaves much more freedom of choice. As a first and well-known example, consider the original Higgs mechanism applied to the electroweak sector of the SM, based on the structure group $G = \mathrm{U}(2) \approx \mathrm{U}(1) \times \mathrm{SU}(2)$. The aim is to hide the original symmetry group G in favour of the residual symmetry H of Maxwell theory

$$G = \mathrm{U}_Y(1) \times \mathrm{SU}(2) \longrightarrow H = \mathrm{U}_{\mathrm{em}}(1) \tag{10}$$

by spontaneous symmetry breaking (SSB), where the $\mathrm{U}_{\mathrm{em}}(1)$ is an appropriate linear combination of the original $\mathrm{U}_Y(1)$ and the one-parameter subgroup generated by the operator $\mathbf{T}_3$ of SU(2). For that purpose one introduces a multiplet of scalar fields Φ, classified by quantum numbers y and (t, t_3) of weak hypercharge $\mathrm{U}(1)_Y$ and weak isospin SU(2), respectively, as well as a G-invariant potential $V(\Phi)$ which exhibits a degenerate minimum at some nonvanishing value ϕ^0. The freedom of choice is reflected by the fact that the multiplet Φ can sit in almost any multiplet of weak isospin $t \geq \frac{1}{2}$ provided it contains one substate $(y, t, t_3^{(\mathrm{H})})$ which is electrically neutral such that it develops a vacuum expectation value $v = \sqrt{(\phi^0, \phi^0)} \neq 0$. The weak hypercharge of "the" Higgs scalar is $y = 1$, its weak isospin is $(t, t_3^{(\mathrm{H})}) = (\frac{1}{2}, -\frac{1}{2})$ so that its electric charge $Q = t_3 + \frac{1}{2}y$ vanishes. However, there is nothing up to this point that would tell us that the Higgs lives in a doublet with respect to SU(2). All one can deduce from the construction of the minimal electroweak SM is the relation (for a derivation of this formula see the appendix)

$$\frac{m_{\mathrm{W}}^2}{m_{\mathrm{Z}}^2 \cos^2\theta_{\mathrm{W}}} = \frac{t(t+1) - t_3^2}{2t_3^2} \tag{11a}$$

which contains three experimental numbers on its left-hand side[2]: the masses of W and Z, and the squared cosine of the Weinberg angle θ_{W}. The derivation of the ratio (11a) is given in the appendix. Strangely enough, it is experiment that tells us that the ratio on the left-hand side of (11a) is equal to 1 within a very small error bar,

$$\left.\frac{m_{\mathrm{W}}^2}{m_{\mathrm{Z}}^2\cos^2\theta_{\mathrm{W}}}\right|_{\mathrm{exp}} = 1.0004\,^{+0.0008}_{-0.0004}\ , \tag{11b}$$

a value which singles out the doublet case. This reminds one of earlier days of nuclear spectroscopy when people determined spins and parities of nuclear excited states by angular correlations. Clearly, it would be more satisfactory if this assignment were a prediction! In fact, as we shall discuss below, extensions of YM theories within noncommutative geometry put the Higgs into the radiation sector and fix its quantum numbers to the values favoured by (11b).
The introduction of fermionic fields such as those describing quarks and leptons also follows standard rules. Like in the example discussed above, there is much, and in fact too much, freedom of choice for the corresponding Dirac operator(s). Let $\Psi(x)$ be a set of fermionic fields classified by a reducible or irreducible representation of G. For definiteness, think of the weak isospin and QCD part of the SM, $G = \mathrm{SU}(2)\times\mathrm{SU}(3)$. The U(1) part is always more problematic than the nonabelian factors because it escapes universality of couplings (Exercise 5). A (still classical) Lagrangian including a scalar multiplet of fields Φ and a fermionic multiplet Ψ which is both globally and locally gauge invariant will have the form

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4q^2\kappa^{(\mathrm{ad})}}\,\mathrm{tr}\ (F_{\mu\nu}F^{\mu\nu}) + \tfrac{1}{2}\,(D_A\Phi, D_A\Phi) - V(\Phi)\\ &+\tfrac{\mathrm{i}}{2}\left(\overline{\Psi}, \gamma\cdot \overleftrightarrow{D}_A\ \Psi\right) - \left(\overline{\Psi}, (\mathbf{M}+\varrho\Phi)\,\Psi\right)\ .\end{aligned} \tag{12}$$

Here, as before, D_A denotes the covariant derivative so that, for example, $\gamma\cdot D_A$ stands for $\gamma^\mu D_\mu(A)$, and $V(\Phi)$ is the Higgs potential,

$$V(\Phi) = \tfrac{\lambda}{4}\left\{(\Phi,\Phi) - v^2\right\}^2 + \mathrm{const.} \tag{13}$$

(The left-right action of the derivative means $f\ \overleftrightarrow{\partial}_\mu\ g = f(\partial_\mu g) - (\partial_\mu f)g$.)
The last term in (12) is not as innocent as it looks at a first glance. The term $\mathbf{M}$ contains the mass matrices of quarks and of leptons, of unknown origin, which are certainly not diagonal in the base states of the representation Ψ of G. The term proportional to the real number ϱ is a Yukawa coupling of the fermions to the Higgs field which will contribute to the fermion masses through the one component ϕ^0 which develops a vacuum expectation value. Of course, either of these terms can be present only if the factors composing them join in a G-invariant manner. For

[2] I am skipping a discussion of radiative corrections which appear in the left-hand expression of (11a) if one uses bare values for the input parameters. This is a standard topic in the discussion of the minimal SM and the rules for including them are well known, see, e.g., the review on the electroweak model [1], p. 125.

instance, in the minimal electroweak SM and with one generation of leptons one has

$$\Psi(x) = \begin{bmatrix} L(x) \\ R(x) \end{bmatrix} , \text{ where } L(x) = \begin{bmatrix} \nu_L(x) \\ e_L(x) \end{bmatrix} \text{ and } R(x) = \begin{bmatrix} e_R(x) \end{bmatrix}$$

are a left-handed doublet with $y = -1$ and a right-handed singlet with $y = -2$. (Verify the electric charges!) As there is no way of constructing a mass term $(\bar{\Psi}, \mathbf{M}\Psi)$ invariant with respect to G, one *must* rely on the Yukawa coupling for giving the electron a mass. This implies that the parameter ϱ, for every fermion, must be tuned such as to yield the empirical mass.

1.2 Mysteries of leptonic interactions

Both the world of quarks and the world of leptons contain a great deal of inner structure which is accessible in experiment but is not understood by any means. Among a longer list of mysteries in this realm we discuss two aspects of special relevance to geometric theories of particle physics.

1.2.1 Chiralities in weak charged-current interactions

One of the great mysteries of lepton physics is the observation that charged weak interactions couple to purely *left-chiral* states only. As long as the three neutrinos were thought to be strictly massless, it was more or less plausible that only neutrino states with *negative* helicity, and antineutrino states with *positive* helicity participated in weak interactions, while states with the opposite helicity did not couple to anything. Helicity plus or minus one-half is an invariant characterization only if the fermion is massless. Furthermore, the occurrence of one state of helicity only is linked to the observed *maximal parity violation* in weak interactions.
For comparison, consider the interaction of photons with charged particles. A good example is the production process

$$e^+ + e^- \longrightarrow \mu^+ + \mu^-$$

via annihilation of the e^+e^--pair into a virtual photon, and the creation of the $\mu^+\mu^-$-pair by annihilation of the same photon. Suppose the colliding electron and positron beams are unpolarized but the orientation of the spins of μ^+ and μ^- along the momenta or opposite to them are recorded. The spin selection rules (Exercise 6) tell us that the chiralities of the positive and the negative muon are correlated: If the μ^+ is right-chiral then the μ^- is left-chiral, but if the μ^+ is left, the μ^- is right. Electromagnetic interactions are strictly parity-conserving that is to say the two chirality constellations couple with the same strength to the intermediate photon state. As a result, the emerging $\mu^+\mu^-$-pair, like the incident e^+e^--pair, will be found to be unpolarized. In fact, the analogous process with τ-leptons,

$$e^+ + e^- \longrightarrow \tau^+ + \tau^- ,$$

is used to produce beams of polarized τ^- by selecting one definite chirality state of its partner τ^+.

In the meantime we have learnt that at least some of the neutrinos do have nonvanishing masses, and, therefore, one might expect that handedness no longer plays a fundamental role. In other terms, models which contain some right-handed weak interactions, besides the dominant left-handed ones, such as left-right symmetric models with appropriate SSB, might describe reality. That this is not so is demonstrated by a beautiful determination of the chirality of the $\bar{\nu}_\mu$. This striking example goes as follows [2]. Muon beams usually stem from charged pion decay, $\pi^- \to \mu^- + \bar{\nu}_\mu$, as illustrated by figure 1 (see also Exercise 7). As this decay is medi-

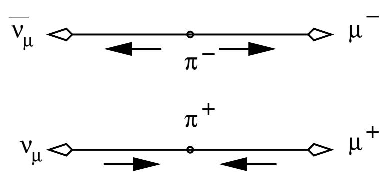

Figure 1: Weak decays of π^- (π^+) into μ^- (μ^+) and $\bar{\nu}_\mu$ (ν_μ)

ated by weak charged interaction which are parity violating, the muon is expected to be longitudinally polarized. Let P_μ be its degree of longitudinal polarization. The negatively charged muon decays predominantly via the process $\mu^- \to e^- \nu_\mu \bar{\nu}_e$. The decay asymmetry of the electron with respect to the muon spin and close to the upper end of the spectrum is calculated to be

$$\left(\frac{\mathrm{d}^2\Gamma}{\mathrm{d}x \, \mathrm{d}\cos\theta} \right) \Bigg|_{x \to 1} = \frac{m_\mu^5 G_\mathrm{F}^2}{144\pi^3} \left\{ 1 - P_\mu \frac{\xi\delta}{\rho} \cos\theta \right\} , \tag{14a}$$

where $x = E/E_{\max}$, θ is the angle between the muon spin and the electron momentum, while ρ, ξ, and δ are real parameters which are calculated from the couplings in a general weak interaction Lagrangian [3] but whose explicit form is of no relevance here. Obviously, the absolute value $|P_\mu \xi\delta/\rho|$ cannot exceed the value 1. The experimental result of Jodidio et al. [4]

$$P_\mu \frac{\xi\delta}{\rho} = 0.9989 \pm 0.0023 \tag{14b}$$

is found to be very close to its maximal value 1. By definition, the longitudinal polarization P_μ cannot be larger than 1. One concludes that both P_μ and the combination $\xi\delta/\rho$ must each lie very close to 1. The result for the former has an immediate consequence for the chirality of the $\bar{\nu}_\mu$ of the antineutrino emitted in pion descay, by conservation of angular momentum. As worked out by Fetscher [2], the experimental result (14b) implies

$$1 - |h(\bar{\nu}_\mu)| < 0.0032 \quad \text{at} \quad 90\% \text{ C.L.} \;. \tag{14c}$$

The signs of $h(\bar{\nu}_\mu)$ and $h(\nu_\mu)$, which are opposite of each other, are known from experiment. Therefore, the result (14c) is convertible to the information

$$h(\bar{\nu}_\mu) = +1 \quad \text{and} \quad h(\nu_\mu) = -1 \;, \tag{15}$$

within very small error bars. This is, by far, the most accurate determination of a neutrino chirality.

1.2.2 Family numbers and selection rules

Another mystery is the separate additive conservation of individual lepton family numbers L_e, L_μ, and L_τ. Soon after the minimal SM of electroweak interactions was developed one realized that it could easily accommodate any number of copies of the (e, ν_e) family as well as arbitrary amounts of state mixing between members of different families. The same statement applies to the quark sector (see below). Here again, the geometry of the SM provides almost no constraint on how fermion multiplets should be added to the bosonic sector, except for the well-known conspiracy in the electric charges and numbers of generations and colors needed to cancel chiral anomalies (see Sect. 3.1 below).
There is a wealth of data supporting the conservation of individual family numbers, see, e.g., the compilation in the review *Tests of Conservation Laws* in [1]. I quote here three prominent examples which show the impressive degree of accuracy to which these conservation laws are known, viz.

$$\frac{\Gamma(\mu^- \to e^- + \gamma)}{\Gamma_{\text{total}}} < \; 1.2 \times 10^{-11} \quad \text{at } 90\% \text{ C.L.} \;, \tag{16a}$$

$$\frac{\Gamma(\mu^- \to e^- + e^+ + e^-)}{\Gamma_{\text{total}}} < \; 1.0 \times 10^{-12} \quad \text{at } 90\% \text{ C.L.} \;, \tag{16b}$$

$$\frac{\sigma(\mu^- \text{Ti} \to e^- \text{Ti})}{\sigma(\mu^-\text{-capture on Ti})} < 4.3 \times 10^{-12} \quad \text{at } 90\% \text{ C.L.} \;. \tag{16c}$$

The first of these compares the rate of a hypothetical "radiative decay" from the muon to the electron with the total decay rate, where the latter is dominated by the allowed process $\mu^- \to e^- + \nu_\mu + \bar{\nu}_e$. Indeed, the eigenvalues of L_e and L_μ in the decay $\mu^- \to e^- + \nu_\mu + \bar{\nu}_e$ and their additive conservation give $(L_e = 0, L_\mu = 1)$ for the left side, and $(L_e = 1 - 1, L_\mu = 1)$ for the right side. The decay $\mu^- \to e^- + \gamma$, in turn, would lead from $(L_e = 0, L_\mu = 1)$ to $(L_e = 1, L_\mu = 0)$ which is forbidden. The decay (16b) differs from the one of (16a) by the photon of (16a) moving off its mass shell and dissociating into an electron-positron pair. The process (16c), finally, compares the cross section for neutrinoless capture of a muon on a nucleus, yielding an electron and the same nucleus in some excited state, with ordinary muon capture where one has a muon neutrino in the final state.

2 Mass matrices and state mixing

In the minimal SM the fermionic states which participate in weak charged-current interactions do not coincide with the eigenstates of mass which couple to the electromagnetic and, in the case of quarks, the strong interactions. Therefore, a certain amount of state mixing becomes observable in weak interaction processes. This phenomenon has been studied in great detail and over many years for the three generations of quarks. Qualitatively the same phenomenon is known in neutrino physics through oscillations of neutrino states, although on a quantitative level things are very different.

2.1 The case of quarks

The quark states with electric charge $+\frac{2}{3}$ and the states with electric charge $-\frac{1}{3}$ which couple to (charge changing) weak interaction vertices are not identical with the states that occur in strong interactions. The states which couple to weak vertices are "rotated" as compared to the mass eigenstates which are the ones coupling to gluons in QCD. The *mixing matrix*, called CKM-matrix (after its discoverers N. Cabibbo, M. Kobayashi, and K. Maskawa), is a unitary 3×3-matrix four parameters of which are observables. The mixing matrix, in the case of quarks, is close to diagonal. In the analogous case of leptons it seems to be far from diagonal. There are few restrictions on the admissible mass matrices in a given charge sector which are to be inserted in their Dirac operator. In particular, they need not be hermitean. We shall assume that they are nonsingular. Once the mass matrices are given, their diagonalization yields the mixing matrix and, of course, the mass eigenvalues,

Mass matrices of *up* and *down* quarks $\Longrightarrow$ CKM-matrix.

That part of the analysis is trivial. What about the converse?
Suppose the mass eigenvalues and the empirical mixing matrix are given. *What is the space of mass matrices which are compatible with these data?* Can one define parameters that help to sweep the space of admissible mass matrices?
The information on the quark masses is the following [1]: The set of *up*-like quarks u_i, charge $q^{(u)} = +\frac{2}{3}$, have the masses

$$m_u = 1.5 \text{ to } 3.3 \text{ MeV} , \; m_c = 1.27 \pm {}^{+0.07}_{-0.11} \text{ GeV} , \; m_t = 171.3 \pm 1.1 \pm 1.2 \text{ GeV} . \quad (17a)$$

The *down*-like quarks d_i, with charge $q^{(d)} = -\frac{1}{3}$, are known to have the masses

$$m_d = 3.5 \text{ to } 6.0 \text{ MeV} \ , \ m_s = 105 \pm {}^{+25}_{-35} \text{ MeV} \ , \ m_b = 4.20\,^{+0.17}_{-0.07} \text{ GeV} \ . \qquad (17b)$$

Regarding the CKM-matrix different conventions for its representation are possible. What is relevant for the problem posed above is the fact that it depends on four real physical parameters, say three mixing angles and one phase (describing CP-violation), all of which are known from experiment.
The key to the inverse analysis is the observation that weak charged-current interactions couple to *left*-chiral fields only. The *right*-chiral fields remain unobservable. In the reconstruction of all mass matrices which are compatible with a given set of data (masses and mixing) one makes use of the complete freedom of choice of right-chiral fields. Furthermore, a simultaneous unitary transformation of the left-chiral fields of charges $+\frac{2}{3}$ and $-\frac{1}{3}$ leaves the observables unchanged. In more detail: Let $M^{(q)}$, $q = u$ and $q = d$, be arbitrary, nonsingular mass matrices for the group of *up*-like and *down*-like quarks, respectively. They are diagonalized by bi-unitary transformations of left- and right-chiral fields,

$$U_L^{(q)} M^{(q)} U_R^{(q)\,\dagger} = \Delta^{(q)} \ , \ \Delta^{(q)} = \text{diag}\ (m_1^{(q)}, m_2^{(q)}, m_3^{(q)}) \quad q = u, d \ . \qquad (18)$$

As is well known, the unitary matrices $U_L^{(q)}$ diagonalize the hermitean, "squared" mass matrices $M^{(q)} M^{(q)\,\dagger}$, while the unitaries $U_R^{(q)}$ diagonalize $M^{(q)\,\dagger} M^{(q)}$. The CKM-matrix is given by the product of the left unitaries in (18), i.e.

$$V_{\text{CKM}} = U_L^{(u)} U_L^{(d)\,\dagger} \ . \qquad (19)$$

The most general transformation of the mass matrices which leaves this matrix invariant reads

$$U^\dagger M^{(q)} V^{(q)} \ , \quad q = u, d = \hat{M}^{(q)} \ , \qquad (20)$$

where U, $V^{(u)}$, and $V^{(d)}$ are arbitrary unitary matrices. Note that the unitary matrices $V^{(u)}$ and $V^{(d)}$ act on *right*-chiral fields and, hence, are independent of each other. The unitary matrix U acts on the *left*-chiral fields and, hence, must be the same in the two charge sectors.
The most economic reconstruction procedure makes use of the polar decomposition theorem for nonsingular matrices [5] (Exercise 8).
Any nonsingular M can be written as a product of a lower- (or upper-)triangular matrix T and a unitary matrix W,

$$M = TW \quad \text{with} \quad T \text{ lower triangular,} \quad W \text{ unitary} \ . \qquad (21)$$

The decomposition is unique up to multiplication of W from the left by a diagonal unitary matrix diag $(\exp\{i\omega_1\}, \ldots, \exp\{i\omega_n\})$. Closer inspection shows the intimate relation of this theorem to Schmidt's orthogonalization procedure well-known, e.g., from quantum mechanics.

Since W acts on right-chiral fields and, hence, is unobservable, the essential information on a given mass matrix is contained in the triangular factor, viz.

$$T^{(u)} \text{ or } T^{(d)} = \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} ,$$

where asterisks denote possibly nonvanishing entries. This form still covers the most general case. Contact to more conventional representations is made by taking the hermitean "squares", $\hat{H}^{(q)} = T^{(q)} T^{(q)\dagger}$ whose eigenvalues are the squared masses,

$$\hat{H}^{(q)} = M^{(q)} M^{(q)\dagger} = T^{(q)} T^{(q)\dagger} = U^{\dagger} \hat{D}^{(q)} U , \tag{22a}$$
$$\hat{D}^{(q)} = \text{diag}\,(m_1^{(q)\,2}, m_1^{(q)\,2}, m_1^{(q)\,2}) .$$

More specifically, by an appropriate choice of basis, one obtains the representations in the *up*- and *down*-sectors

$$\hat{H}^{(q+1)} = U^{\dagger} \hat{D}^{(q+1)} U , \qquad \text{(up sector)} , \tag{22b}$$
$$\hat{H}^{(q)} = U^{\dagger} V_{\text{CKM}} \hat{D}^{(q)} V_{\text{CKM}}^{\dagger} U , \qquad \text{(down sector)} . \tag{22c}$$

As we showed earlier [6] the matrix U is known *analytically.* Given the masses and the mixing matrix the remaining freedom in choice of the unitary U is contained in two complex parameters which, in turn are constrained by a *quadratic* equation [7]. That is to say, the freedom eventually reduces to *one* complex parameter (or two real parameters). While this parameter runs through its domain of definition (bounded by a circle with radius $R = \sqrt{(m_t^2 - m_u^2)/(m_t^2 - m_c^2)}$ about the origin) one sweeps the space of admissible quark mass matrices.

Remark: Earlier analyses such as [8] made use of what was called the nearest-neighbour interaction (NNI) by assuming, from the start, the mass matrices to have the form

$$\hat{M} = \begin{pmatrix} 0 & * & 0 \\ * & 0 & * \\ 0 & * & * \end{pmatrix} .$$

There were two intuitive physical ideas behind this ansatz:
(i) initially, before interactions are switsched on, only the third generation has a nonvanishing mass;
(ii) only immediate neighbours are allowed to interact.

Unfortunately intuition was misled, this picture is ill-defined because *any* set of matrices $M^{(q+1)}$, $M^{(q)}$ can be brought to NNI form, just by choosing the bases of chiral states appropriately. In other words, the NNI representation still covers the most general case. Furthermore, if the same ansatz is converted to the triangular representation by the decomposition Eq. (21) then one sees that now it is the $i = 2, k = 1$-element that vanishes, $\hat{T}_{21}^{(q)} = 0$. This certainly is counter-intuitive because it might suggest that there is no direct interaction between the first and the second generations, while in the first ansatz they seemed to interact strongly.

Since then, Häußling found an alternative procedure [9] which is technically much simpler than our earlier analysis [7]. He shows that the unobservable right-chiral fields can be chosen such that one arrives at the following representation:

$$\Delta^{(u)} V_{\mathrm{CKM}} \Delta^{(d)} = \hat{M}^{(u)\,\dagger} \hat{M}^{(d)} \ , \tag{23a}$$

$$\hat{M}^{(u)} = V \Delta^{(u)} \quad \text{and} \quad \hat{M}^{(d)} = \hat{M}^{(u)} \Delta^{(u)\,-1} V_{\mathrm{CKM}} \Delta^{(d)} \ , \tag{23b}$$

$$\hat{M}^{(d)} = \hat{M}^{(u)} \Delta^{(u)\,-1} V_{\mathrm{CKM}} \Delta^{(d)} \ . \tag{23c}$$

Here the mass matrices $\hat{M}^{(q)}$, $q = u, d$, have rows whose squared norm is $m_{u_i}^2$ or $m_{d_i}^2$, respectively, while any two different rows are orthogonal to each other. The matrix V is an arbitrary unitary. Note that the left-hand side of (23a) contains the experimental input. Thus the product $\hat{M}^{(u)\,\dagger} \hat{M}^{(d)}$ can be determined from experiment. By varying the unitary V one reaches all mass matrices compatible with experiment. Eq. (23c), finally, yields a linear relationship between the *up* and the *down* sector. The "constant of proportionality" is the experimental input $\Delta^{(u)\,-1} V_{\mathrm{CKM}} \Delta^{(d)}$.
A few comments on these results are the following

1. All formulae in the analysis [7] are explicit, fairly simple, and can be studied in a transparent manner as a function of the one complex parameter on which they depend. This appears to be the best one can do in reconstructing the mass matrices from the data (mass eigenvalues and observed mixings).

2. That parameter represents, so to speak, "the heart of the matter". The NNI representations are the most rational representations. Therefore, any specific model that is proposed for the quark mass matrices can be tested by converting them to that class of bases and checking for compatibility.

3. The more recent analysis by Häußling [9], of course, is compatible with the NNI representation but presents the advantage of reconstructing *all* mass matrices, up to equivalence due to allowed but unobservable unitary transformations. To quote an analogy: the one-parameter NNI setting is like singling out a specific representative state ψ of a quantum mechanical ray, while the general method yields the whole ray $\{\exp \mathrm{i}\alpha\psi\}$. This is true as long right-chiral fermion fields remain unobservable in weak interactions.

4. Matters would change immediately if one discovered additional interactions which were sensitive to left- and right-chiralities in a physically relevant way. Then the "freedom of phases" decribed in the preceding remark, is lost.

5. It is a matter of convention whether one takes the mass matrix of up-type quarks to be diagonal and assumes that the mixing occurs in the *down*-sector only. Our analysis above shows that the mixing can be shifted to either one of the two charge sectors, or be distributed over both.

2.2 The case of leptons

Everything that was said about quarks in Sect. 2.1 also holds for the three generations of leptons

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix} , \quad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix} , \quad \text{and} \quad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix} . \tag{24}$$

In defining the charge -1 and charge 0 states that participate in (charge changing) weak interactions, with reference to the mass eigenstates, one is free to assume neutrino states to be mixed, or electron-like states to be mixed, or even a combined configuration where both charge sectors mix. The electromagnetic and weak *neutral* interactions are diagonal in the flavours and, therefore, are insensitive to mixing described by unitary matrices. Unfortunately, although the masses are better defined than for quarks the experimental information on neutrino masses and mixing is much more scarce. The present state of knowledge is as follows [1] (for a more detailed discussion see the review [10]):

$$\begin{aligned}
&\Delta m^2_{21} = (7.59 \pm 0.20) \times 10^{-5} \text{ eV}^2 , \\
&\sin^2(2\theta_{12}) = 0.87 \pm 0.23 , \\
&\Delta m^2_{32} = (2.43 \pm 0.13) \times 10^{-3} \text{ eV}^2 , \\
&\sin^2(2\theta_{23}) > 0.92 , \\
&\sin^2(2\theta_{13}) < 0.19 , \text{ at C.L.} = 90\% .
\end{aligned} \tag{25}$$

As such, these data are not sufficient for an analysis of the kind discussed above. Nevertheless, there are certain patterns as well as specific models that were proposed by various people (including the autor) and it might be worthwhile to check them for internal consistency in the general framework developed for quarks.

3 The space of connections and the action functional

As we emphasized previously in Subsect. 1.1, although meant to describe *quantum* field theories, the construction of nonabelian gauge theories seems to be a purely *classical* construction. The remaining three sections will show that this is not really true for several reasons. First, one can show that not every classical YM theory, after quantization, becomes a viable theory. Second, it may be that YM theories are embedded in the more restrictive framework of noncommutative geometry so that, again, not every structure group can be "gauged" and be converted to an acceptable quantum YM theory.

3.1 The axial anomaly, a reminder

A class of obstructions that were known already very early through the work of Adler, Bell, Jackiw, and others, concern local and global *anomalies*. For example, the renormalizability of the minimal electroweak SM is threatened by the triangle anomaly involving an axial current. The axial vector part of the fermionic current

$$a_\mu = \sum_{f=e,\mu,\tau} \overline{\Psi_f(x)}\gamma_\mu\gamma_5 U(Y)\Psi_f(x) + \sum_{q=1}^{3}\sum_{c=1}^{3} \overline{\Psi_{q,c}(x)}\gamma_\mu\gamma_5 U(Y)\Psi_f(x)q,c \quad (26)$$

which couples to the gauge field $A_\mu^{(0)}(x)$, produces an anomaly in its divergence which is proportional to

$$S = S_{\text{leptons}} + S_{\text{quarks}} \quad \text{with}$$
$$S_{\text{leptons}} = \sum_{e,\mu,\tau} \text{tr}\ \{U(T_m T_m Y)\} \text{ and } S_{\text{quarks}} = \sum_{q,c} \text{tr}\ \{U(T_m T_m Y)\}\ , \quad (27)$$

where T_m is a component of weak isospin and U denotes the respective fermion representations. The sums run over the three generations of leptons and quarks, and, in the case of the quarks, over the colour quantum number. Note that only isospin doublets contribute and that it is sufficient to consider the component $m = 3$ only. Then, marking the factors three from flavour and from colour, one finds

$$S_{\text{leptons}} = 3_f \cdot \frac{1}{4} \cdot (-2) = -\frac{3}{2}\ ,\ S_{\text{quarks}} = 3_f \cdot \frac{1}{4} \cdot 3_c \cdot \frac{2}{3} = \frac{3}{2}\ , \quad (28)$$

whose sum vanishes indeed. Thus, the electroweak SM is safe only if there is this conspiracy between the lepton families and the quark multiplets. We note in passing that the factor three which stems from the colour degrees of freedom, is essential in explaining the absolute magnitude of the amplitude for $\pi^0 \to \gamma\gamma$ decay.

3.2 Geometric route to anomalies

It is well-known that anomalies occur at the order $\hbar$, i.e. at the level of one-loop diagrams. This may be the reason why they can be identified also within the geometric approach [11, 12], beyond the algebraic analysis sketched above. Without going into details let me describe the essence of this approach by the following somewhat sketchy remarks. A more detialed account can be found in the papers quoted above and in [13].
A given YM theory is formulated on a principal fibre bundle

$$\mathcal{P} = \left(P \stackrel{\pi}{\to} M, G\right)\ . \quad (29)$$

Both the gauge group $\mathfrak{G}$ which is the group of vertical automorphisms on $\mathcal{P}$, and the space $\mathfrak{A}$ of connections on $\mathcal{P}$ are infinite dimensional. The space of connections is an affine space and, hence, is mathematically simple. From the point of view of physics, the space $\mathfrak{A}$ contains by far too much freedom. Physics can only depend on gauge potentials which are not gauge equivalent. So, roughly speaking, the space of connections should be divided into classes of gauge-equivalent connections and only these classes should appear in the action. Now, the action of the gauge group $\mathfrak{G}$ on $\mathfrak{A}$, in general, is highly nontrivial so that

$$\mathcal{M} := \mathfrak{A}/\mathfrak{G} \quad (30)$$

is a rather complicated object which, in general, is not a manifold. In other words, brute-force division by the gauge group is dangerous and, perhaps, not even possible. The action functional, integrated over the fermions, is formulated on $\mathfrak{A}$. On the other hand, the effective functional must depend only on the gauge-inequivalent connections.

A more gentle way of performing this division is to do it, if possible, stepwise. For that purpose one studies the stratification of $\mathfrak{A}$ by the action of the gauge group, so that it is decomposed as follows

$$\mathfrak{A} = \mathfrak{A}^{(J_0)} \cup \mathfrak{A}^{(J_1)} \cup \cdots \mathfrak{A}^{(J_k)} , \tag{31a}$$

where the individual stratum is characterized by the stability group $\mathfrak{G}_A$ of its elements A being conjugate to J_i, $i = 0, \ldots, k$. That is to say

$$\mathfrak{A}^{(J_i)} = \left\{ A \in \mathfrak{A} | \, \mathfrak{G}_A = \psi J_i \psi^{-1} \, , \psi \in \mathfrak{G} \right\} . \tag{31b}$$

Here J_i is isomorphic to a subgroup of the structure group G. The number of strata is countable. The main stratum (index J_0) is characterized by J_0 isomorphic to the center C of G. The stratification (31a) is unique and natural once the framework is defined by giving $\mathcal{P}(M, G)$. The formal definition

$$\mathcal{M}_i = \mathfrak{A}^{(J_i)}/\mathfrak{G}$$

yields an orbit bundle decomposition which bears some similarity to the compactification procedure in Kaluza-Klein theories. The spaces $\mathcal{M}_i$ are parts of the space of physical, gauge-inequivalent connections. The trouble is that, in general, they cannot be joined to a smooth manifold.

Physics comes in via an action functional $S(A, \bar{\psi}, \psi)$ which is a classical functional and is strictly gauge invariant. At the quantum level the central quantity is the generating function

$$Z(A) = \int [\mathcal{D}\chi][\mathcal{D}\bar{\chi}] \exp\{-(S + \bar{\chi} \, \partial\!\!\!/_A \, \chi)\} \tag{32}$$

obtained while integrating the fermionic degrees of freedom. There are two possibilities. Either $Z(A)$ is strictly invariant under a gauge transformation ψ, $Z(\psi A) = Z(A)$, or it is equivariant but not strictly invariant, $Z(\psi A) = \varrho^{-1}(A, \psi) Z(A)$, where ϱ is the action of the gauge transformation. In the first case one can safely divide by the gauge group to obtain a perfectly acceptable functional. The theory has no anomaly and can be reduced to the space $\mathcal{M}$ of gauge-inequivalent connections. In the second case, in contrast, there must be anomalies, the reduction is not possible. Therefore, in the geometric framework anomalies are obstructions to the reduction procedure which are due to quantization.

Without going into further, mostly technical details, let me sketch a constructive way of identifying anomalies in this geometric setting. One well-known ansatz is to make use of what is called the *pointed gauge group* $\mathfrak{G}^*$. This is the subgroup of $\mathfrak{G}$ which is the stability group of an arbitrary but fixed point p_0 of the principal fibre bundle $\mathcal{P}(M, G)$,

$$\mathfrak{G}^* = \mathfrak{G}_{p_0} = \{ \psi \in \mathfrak{G} | \, \psi(p_0) = p_0 \} . \tag{33}$$

In other terms the pointed gauge group acts like the identity in the fibre over p_0. Singer showed that the action of $\mathfrak{G}^*$ on $\mathfrak{A}$ is free [14]. Therefore,

$$\begin{array}{ccc} \mathfrak{G}^* & \longrightarrow & \mathfrak{A} \\ & & \downarrow \\ & & \mathcal{M}^* = \mathfrak{A}/\mathfrak{G}^* \end{array}$$

is a principal fibre bundle. The functional $Z(A)$ is a trivial section

$$Z \,:\, \mathfrak{A} \longrightarrow \text{Det} \;:= \mathfrak{A} \times \mathbb{C} \tag{34a}$$

in the determinant bundle. If one divides by the pointed gauge group one obtains the reduced section

$$Z^* \,:\, \mathfrak{A}/\mathfrak{G}^* \longrightarrow \left(\mathfrak{A} \times \mathbb{C}\right)/\mathfrak{G}^* =: \text{Det}^{\,*} \,. \tag{34b}$$

If $Z(A)$ is strictly invariant then the action of $\mathfrak{G}^*$ on $\mathbb{C}$ is trivial so that (34b) reduces to

$$Z^* \,:\, \mathcal{M}^* \longrightarrow \mathcal{M}^* \times \mathbb{C} \,, \quad \mathcal{M}^* = \mathfrak{A}/\mathfrak{G}^* \,. \tag{34c}$$

In turn, if $Z(A)$ is equivariant but not strictly invariant then the action of the pointed gauge group on $\mathbb{C}$ is not trivial. In this situation Det^* has a twist, the integration over $[\mathcal{D}A]$ is not possible. Geometrically speaking there is a topological anomaly.
Even if no such anomaly is encountered, the story is not finished. There remains the "division" by the remainder $\mathfrak{G}/\mathfrak{G}^*$ which is isomorphic to the structure group G. This last step is particularly important because it is the structure group which defines the conserved charges of the theory. Again, if Z^* is strictly invariant the final division poses no problem. If it is not but is (only) equivariant one obtains

$$Z^{**} \,:\, \mathcal{M} \longrightarrow \left(\mathcal{M}^* \times \mathbb{C}\right)/G =: \text{Det}^{**} \,, \tag{35}$$

the functional Z^{**} is nontrivial, and one has found an anomaly.
In summary, by following this geometrical method one identifies all topological as well as possible global, nonperturbative anomalies. More on this can be found in the references given above.

4 Constructions within noncommutative geometry

The reconstruction of the minimal SM as well as of more general gauge and gravitational theories by means of noncommutative geometry is of geometrical origin but goes far beyond the classical framework of local gauge theories. The class of admissible gauge groups is restricted, spontaneous symmetry breaking (SSB) occurs as a rather natural phenomenon, and at least part of what we observe in the matter sector obtains a geometrical backbone. There are essentially three lines of proceeding that were explored extensively:

(i) The construction of the action by means of Dirac operators as advocated by A. Connes and his collaborators,

(ii) The somewhat more empirical construction within the Mainz-Marseille model, and

(iii) the numerous programs of formulating quantum field theory on noncommutative spaces as pioneered by Madore, Grosse, Wulkenhaar, and others.

I do not talk about the third group which, in fact, would cover a series of lectures of its own. Instead, I briefly highlight the constructions (i) and (ii) but without going into much detail.

4.1 Spectral triples and all that

The original construction of the standard model by A. Connes and J. Lott [15] lead rather naturally to SSB with a Higgs potential whose parameters were functions of the quark masses. Thus, for a while it seemed as though the essential parameters of the minimal SM (Weinberg angle, Higgs mass) could be predicted. This, however, was not successful [16]. Furthermore, the model struggled with the correct assignments regarding additive quantum numbers. A much more ambitious approach was proposed later by A. Chamseddine and A. Connes [17] by postulating the Spectral Action Principle. In both approaches the Dirac operator D plays the central role. The spectral action principle asserts that the operator D is all that is needed to define the bosonic part of the action. Since the disjoint union of spaces corresponds to direct sums of Dirac operators, the action functional - which is determined by D - must be additive and, hence, must have the form

$$S = \mathrm{tr}\ \left(f(D/\Lambda)\right)\ , \tag{36}$$

where f is an even function of its real variable, and Λ is a parameter which fixes the mass scale. The theory is determined by a *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ containing a $*-$algebra $\mathcal{A}$, a Hilbert space $\mathcal{H}$, and a Dirac operator D, $\mathcal{A}$ and D being represented on Hilbert space. Of course, matters are not as simple as that. In fact the spectral *triple* rather is at least *quintet* because further data are needed to define it. The simplest realistic example, in terms of physics, is

$$(\mathrm{C}^\infty(M), L_2(M,S), \partial\!\!\!/)\ , \tag{37}$$

with M a compact oriented spin manifold. Here the algebra is the algebra of smooth functions on M and is commutative. The Dirac operator reduces to the ordinary partial derivatives. In view of the SM, in turn, one chooses the data to be

$$\mathcal{A} = \mathrm{C}^\infty(M) \otimes \mathcal{A}_F\,, \tag{38a}$$
$$\mathcal{H} = L_2(M,S) \otimes \mathcal{H}_F\,, \tag{38b}$$
$$D = \partial\!\!\!/ \otimes \mathbb{1}_F + \gamma^5 \otimes D_F\ . \tag{38c}$$

The algebra $\mathcal{A}_F$ is finite and is chosen to be [18]

$$\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_3(\mathbb{C}) \ . \tag{39}$$

A connection is introduced into this model by replacing D by the covariant operator

$$D \longrightarrow D_A = D + A + JAJ^{-1} \tag{40}$$

Implementing charge conjugation properly - and this applies to all NC models - needs special consideration as was first noted in [19]. The operation J, called reality structure, which was introduced later, does this job in Connes' framework. With the choices (38a)–(38c) the connection is

$$A = \gamma^5 \otimes \Phi - \mathrm{i}\gamma^\mu \otimes A_\mu \,, \tag{41}$$

where Φ is a scalar field on M which takes values in $\mathcal{A}_F$, whereas A_μ pertains to a one-form which takes its values in the Lie algebra Lie $(U(\mathcal{A}_F))$. (Note that the structure group must be found in a unitary subalgebra of $\mathcal{A}_F$. This limits the class of structure groups which one can reproduce in this way.)
Of course, there is much more to be said about this fascinating theoretical framework. It provides an interesting ansatz for combining YM theories with gravity and there are branches of it exploring various directions. Its weakness, from a physicist's point of view, is the Euclidean framework. Although there were attempts to generalize spectral triples to Minkowski signature, there still are no satisfactorey answers. Our short excursion may be sufficient to illustrate our main assertion: noncommutative geometry brings in *more* structure into gauge theories of fundamental interactions.

4.2 The bosonic sector à la Mainz-Marseille

The Mainz-Marseille model is a more heuristic construction but, I claim, is closer to phenomenology because it is formulated on Minkowski space with the right causal signature from the start and because it contains less freedom than other models. Instead of a detailed exposition of the model and of what it can do and what not, I illustrate its salient features by three items.
The model is based on a bi-graded differential structure by composing the exterior algebra on M^4 (Minkowski space) and a graded Lie algebra akin to the electroweak structure group U(2) [19], [20]. In the minimal case, the graded algebra is chosen to be

$$\mathrm{SU}(2|1) = \left\{ \mathbf{M} |\, \mathbf{M}^\dagger = -\mathbf{M} \,, \mathrm{Str}\, \mathbf{M} = 0 \right\} \ , \tag{42}$$

the symbol Str denoting the super-trace. In the defining representation these matrices have the form

$$\mathbf{M} = \left(\begin{array}{cc|c} * & * & * \\ * & * & * \\ \hline * & * & * \end{array} \right) \equiv \left(\begin{array}{c|c} \mathbf{A}_{2\times 2} & \mathbf{C}_{1\times 2} \\ \hline \mathbf{D}_{2\times 1} & \mathbf{B}_{1\times 1} \end{array} \right) . \tag{43}$$

The blocks along the diagonal are *even* with regard to the algebra grading in (42), the blocks which sit off the main diagonal, are *odd*. (In this representation the super-trace is Str $\mathbf{M} = \mathrm{tr}\,\mathbf{A} - \mathbf{B}$.) In terms of generators of the graded Lie algebra (42) the ones in the even part must be the T_k of SU(2) and T_0 (or Y) of the U(1) factor. The odd generators, in turn, sit in the off-diagonal blocks.
A rather natural way of constructing a connection for this model is to put ordinary gauge fields into the diagonal blocks of (43) along with the even generators of SU(2|1). If one takes the bi-grading seriously then the connection should have total grade 1 where the total grade is the sum of the exterior form grade and of the internal (matrix) grade. This suggestive structure invites one to fill the odd blocks in the connection by fields which must be zero-forms (with regard to the exterior algebra), one-forms within the algebra and, last not least, doublets with respect to weak isospin. In other terms, one obtains an isospin doublet, scalar field just the way the empirical SM had imposed on us. Conversely, if the Higgs field were not a doublet, it would not fit into the connection, this whole picture would fail.
If one then sits down and works out the Lagrangian of this model [20] one finds the SM Lagrangian with a Higgs doublet which sits in the right (shifted) SSB phase and a potential $V(\Phi)$ which has the correct shape. Thus, SSB is an unavoidable consequence of the model! Schematically and without repeating a detailed calculation, this may be understood as follows.
Generally, representations of SU(2|1) are characterized by two quantum numbers, I_0 (denoted such because it is parent of weak isospin) and Y_0 (parent of weak hypercharge). As shown by Marcu [21] and described in a book by Scheunert [22], the *adjoint representation* of SU(2|1) has the quantum numbers $[I_0 = 1, Y_0 = 0]$, and decomposes in terms of SU(2)×U(1) as follows

$$\begin{aligned}&[I_0 = 1, Y_0 = 0] \longrightarrow (I = 1, Y = 0) \oplus (I = 0, Y = 0)\\ &\oplus (I = \tfrac{1}{2}, Y = 1) \oplus (I = \tfrac{1}{2}, Y = -1)\ .\end{aligned} \tag{44}$$

Note that this is precisely what one needs: A triplet of gauge bosons with vanishing weak hypercharge, a singlet with vanishing hypercharge, and two doublets with $Y = \pm 1$ for the "standard" Higgs fields!
The model also suggests more structure in the *fermionic* sector of the SM. In the case of quarks, each of the three *quark* generations is classified by the simplest *typical representation* of SU(2|1) which reads, together with its decomposition in terms of SU(2)×U(1),

$$\begin{aligned}&\left[I_0 = \tfrac{1}{2}, Y_0 = \tfrac{1}{3}\right] \longrightarrow\\ &(I = \tfrac{1}{2}, Y = \tfrac{1}{3}) \oplus (I = 0, Y = \tfrac{4}{3}) \oplus (I = 0, Y = -\tfrac{2}{3})\ .\end{aligned} \tag{45}$$

Furthermore, SU(2|1) possesses reducible but indecomposable representations where these generations are joined by semi-sums in the following way:

$$\left[I_0 = \tfrac{1}{2}, Y_0 = \tfrac{1}{3}\right] \ni\!\!\!\!+ \left[I_0 = \tfrac{1}{2}, Y_0 = \tfrac{1}{3}\right] \ni\!\!\!\!+ \left[I_0 = \tfrac{1}{2}, Y_0 = \tfrac{1}{3}\right]\ . \tag{46}$$

In this representation the generators have block triangular form, just like the (lower) triangular mass matrices considered earlier, see Eq. (21).

Likewise, one generation of *leptons* fits into the *fundamental* representation of SU(2|1),

$$\left[I_0 = \tfrac{1}{2}, Y_0 = -1\right] \to (I = \tfrac{1}{2}, Y = -1) \oplus (I = 0, Y = -2) \; . \tag{47}$$

Here again, identical representations of this kind may be joined in a semi-direct sum analogous to (46) such as to describe the three lepton families.
It is tempting to use these reducible but indecomposable representations of SU(2|1) for classifying leptons and quarks [19], [23]. If one does so one discovers a scheme which does not fix absolute parameters but reveals certain textures in the mass matrices which are in agreement with experiment.
Finally, the graded structure of the Mainz-Marseille model also fits very well with the quantum number assignment of quarks and leptons, with the absence of anomalies, and with charge quantization [24].

5 Further routes to quantization via BRST symmetry

Spontaneous symmetry breaking within the SM and its extensions remains a puzzle. Although the *classical* geometric setting for describing SSB from the original symmetry group G to the residual symmetry H,

$$G \longrightarrow H \; ,$$

in relation to Goldstone's theorem, looks convincing, it is not at all clear whether nature has chosen this route. Here again, and in view of the forthcoming searches at the Large Hadron Collider (LHC) it might be instructive to first consider the present experimental situation.

5.1 General remarks on SSB and experimental information

The standard model is a renormalizable quantum field theory. Thus, it allows to calculate radiative corrections to an impressive accuracy. However, radiative corrections applied to a specific observable at a given energy scale always receive contributions from other parts of the theory, including contributions from constituents which, for kinematical reasons cannot be seen (yet) at the scales in question. A striking example is the *top*-quark t whose mass was deduced from radiative corrections, to a fair accuracy, before it actually was discovered. To get a feeling for analyses of this kind let me quote a global fit to all observables within the SM published recently by the Gfitter group [25]. Excluding the directly measured and by now well known mass of the top from the fit, radiative corrections alone yield $m_t = 178.2\,^{+9.8}_{-4.2}$ GeV which is not far from the value $m_t = 172.4 \pm 1.2$ GeV obtained from experiment.
The same fit when used to predict the Higgs mass m_H, is less conclusive. There is an experimental lower limit of the order of 114 GeV from LEP and Tevatron experiments. If one excludes that constraint one obtains

$$M_H = 80\,^{+30}_{-23} \text{ GeV} \; . \tag{48}$$

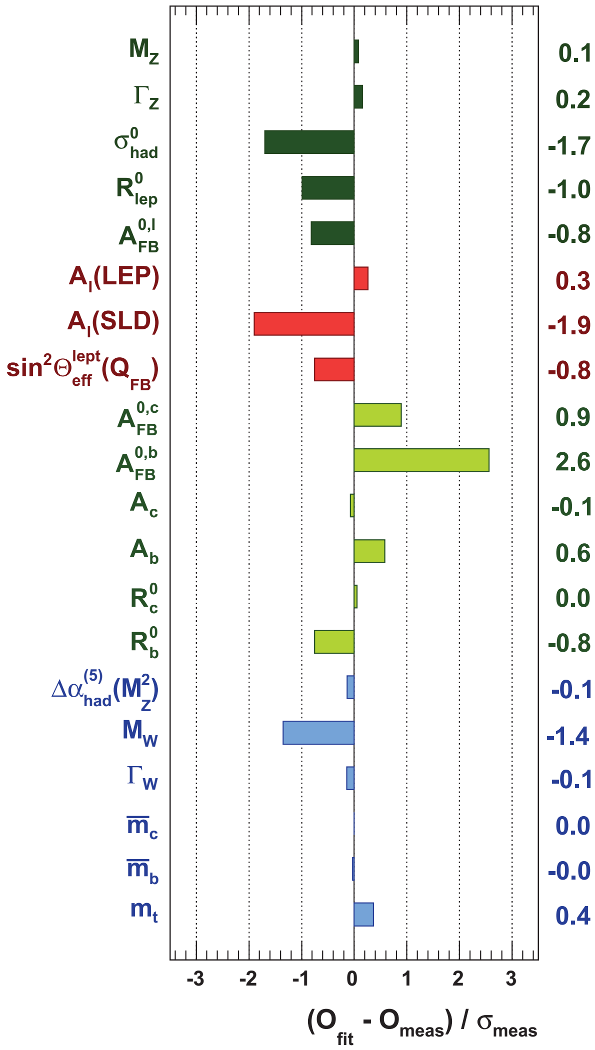

Figure 2: Pull values of observables in a *complete* fit to electroweak observables (taken from [25])

The complete fit, including the mass limits, yields

$$M_H = 116.4\,^{+18.3}_{-1.3}\ \text{GeV}\ . \tag{49}$$

Figure 2 shows the *pull values* for the complete fit, the pull value of an observable being defined by

$$\frac{1}{\sigma|_{\text{meas}}}\left(\mathcal{O}|_{\text{fit}} - \mathcal{O}|_{\text{meas}}\right)\ ,$$

with $\sigma|_{\text{meas}}$ the error in the measurement. The hadronic asymmetry into b-quarks yields a tendency to rather high values of the Higgs mass while the leptonic asymmetries either agree with the overall fit or would prefer an even lower value than that. Obviously, the situation is much less clear than it was for the top before its discovery. This is further illustrated by the following two figures: Figure 3 shows the results for the Higgs mass that one obtains if all sensitive observables are excluded from the standard fit, except the one indicated. This information is complemen-

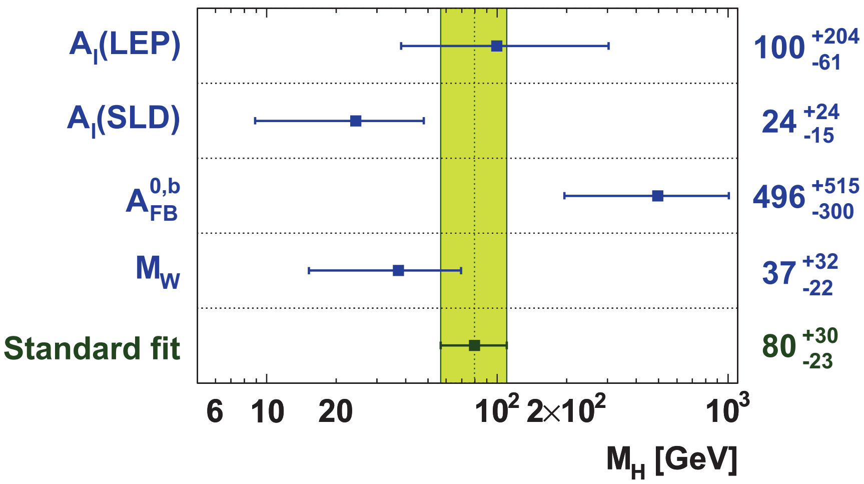

Figure 3: Results for m_H from standard fit excluding the respective measurements (taken from [25])

tary to the information given in figure 4 which shows the results for m_H obtained from the standard fit but excluding the respective measurements.
Various alternatives were discussed in the literature including variations of the minimal SM (two doublet Higgs models, or more, as well as other scenarios).

5.2 A simple model

Scharf was among the first to point out that SSB need not be based on the traditional Higgs mechanism but could be derived as a consequence of Causal Gauge Invariance (CGI) [26]. Causal gauge invariance is a systematic method of treating perturbative gauge theories in the framework of regularization and renormalization developed by Epstein and Glaser [27]. Scharf developed it originally for quantum electrodynamics [28] but it was also applied successfully to massive non-Abelian theories [29]. The instructive example of an Abelian theory was worked out in quite some detail in [31].
We are presently reexamining this route from various points of view [32] in a paper that we hope to publish soon. The following simple model which is taken from this reference, will help to understand some of the ideas which guide one in this approach. Let $\vec{\Phi}$ be a doublet of scalar fields, regarded as a complex singlet $\Phi = v+\varphi+\mathrm{i}B$ where v is a constant (a "vacuum expectation value" in the standard picture), B is a Stückelberg field whose role is to give mass to vector bosons of the model, and φ is a scalar field. The model contains a Higgs Lagrangian,

$$\mathcal{L}_\Phi = \frac{1}{2}\left(\partial_\mu + \mathrm{i}gA_\mu\right)\Phi^\dagger\left(\partial^\mu + \mathrm{i}gA^\mu\right)\Phi + \frac{1}{2}\mu^2\Phi^\dagger\Phi - \frac{1}{4}\lambda\left(\Phi^\dagger\Phi\right)^2 . \tag{50a}$$

As it contains a massive vector boson A whose mass is $m = gv$, the model needs

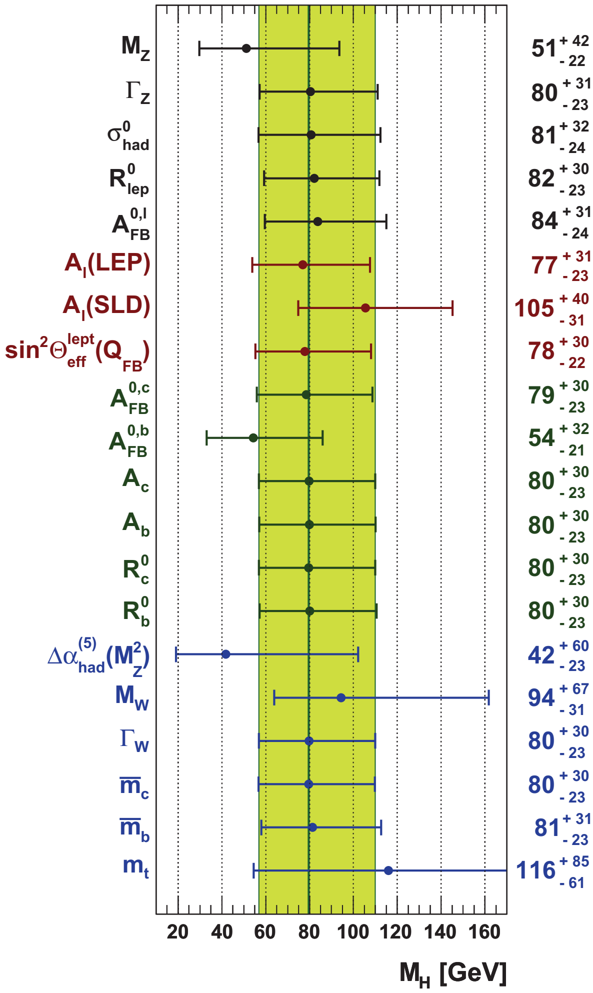

Figure 4: Results for m_H from standard fit excluding the respective measurements (taken from [25])

a gauge fixing term,

$$\mathcal{L}_{\text{g.f.}} = -\tfrac{1}{2}\Lambda\left(\partial_\mu A^\mu + \tfrac{m}{\Lambda}B\right)^2 , \tag{50b}$$

where the constant Λ determines the gauge, ($\Lambda = 1$ is the Feynman gauge). Finally, it is accompanied by a ghost Lagrangian

$$\mathcal{L}_{\text{gh}} = (\partial_\mu \tilde{u}) \cdot sA^\mu - \tfrac{m}{\Lambda}\tilde{u}\, sB \equiv \mathcal{L}^{(0)}_{\text{gh}} + \mathcal{L}^{(1)}_{\text{gh}} , \tag{50c}$$

where $\tilde{u}$ is the antighost field, s is the BRS-operator. The action of the BRS operator on the fields of the model is

$$sA^\mu = \partial^\mu u , \tag{51a}$$
$$su = 0 , \quad s\tilde{u} = -\left(\Lambda\partial_\mu A^\mu + mB\right) , \tag{51b}$$
$$s\Phi = \mathrm{i}gu\Phi \quad \text{or} \quad sB = mu + g\,u\varphi , \quad s\varphi = -gBu . \tag{51c}$$

The gauge fixing term can be rewritten by means of (51b),

$$\mathcal{L}_{\text{gf}} = \tfrac{1}{2}\left(\partial_\mu A^\mu + \tfrac{m}{\Lambda}B\right)(s\tilde{u}) .$$

One then shows that s when applied to $\mathcal{L}_\Phi$ gives zero, $s\mathcal{L}_\Phi = 0$ (Exercise 9). Regarding the other two terms (50b) and (50c) one shows that

$$s\left(\mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{gh}}\right) = \partial_\mu\left((s\tilde{u})(sA^\mu)\right)$$

is a total divergence (see same Exercise). The nilpotency of s is easy to check. One has $(s \circ s)\,\Phi = 0$ while $(s \circ s)\,\tilde{u}$ vanishes for solutions of the field equations. The field equations of the vector field A and of the Stückelberg field are seen to be

$$\left(\Box + m^2\right) A^\mu = (1-\Lambda)\partial^\mu(\partial_\nu A^\nu) , \tag{52}$$
$$\left(\Box + \frac{m^2}{\Lambda}\right) B = 0 . \tag{53}$$

If a gauge other than the Feynman gauge is chosen, i.e. if $\Lambda \neq 1$, the mass of the Stückelberg field is $\frac{m}{\sqrt{\Lambda}}$. Working out the Lagrangian in terms of the field degrees of freedom one obtains

$$\begin{aligned}\mathcal{L}^{(0)} = {} & \mathcal{L}^{\text{YM}}_{\text{kin}}(A) + \tfrac{1}{2}m^2 A_\mu A^\mu - \Lambda\tfrac{1}{2}\left(\partial_\mu A^\mu\right)^2 + \partial_\mu\tilde{u}\,\partial^\mu u - \tfrac{m^2}{\Lambda}\tilde{u}u \\ & + \tfrac{1}{2}\left(\partial_\mu B \partial^\mu B\right) - \tfrac{m^2}{2\Lambda}B^2 + \tfrac{1}{2}\left(\partial_\mu\varphi\,\partial^\mu\varphi\right) - \tfrac{1}{2}m_H^2\varphi^2 - m\partial_\mu\left(A^\mu B\right) .\end{aligned} \tag{54}$$

The individual contributions to this Lagrangian are easy to identify: The first term is the YM "kinetic" term, the second term is the mass term of the vector boson. The third comes from the gauge fixing (50b), the fourth and fifth terms come from the ghost Lagrangian (50c). The two contributions that follow pertain to the B-field, followed by kinetic and mass terms of the scalar field φ. In a similar way one works out the interaction terms at order g and order g^2, see [32], and verifies that the total Lagrangian is consistent with BRST-symmetry, $s(\mathcal{L}^{(0)} + \mathcal{L}_{\text{int}}) = \partial_\mu I^\mu$. What the model shows is this: The CGI-approach yields structures which look very much alike the ad-hoc Higgs mechanism without assuming a potential with a degenerate minimum ab initio. The BRS-symmetry is instrumental in this construction.

6 Some conclusions and outlook

Starting from classical field theory, the basic structure of gauge theories seems to distinguish radiation from matter as two categories which, a priori, have little to do with each other. As soon as one enters the quantum world, however, the distinction can no longer be maintained. Quarks and leptons are described by a Dirac operator which, in turn, is the driving force in the construction of noncommutative geometries designed to generalize Yang-Mills theory. It is not known to which extent the realistic Dirac operator, with regard to the complexity of its mass sectors, determines the (NC) geometry on which quantum field theories for the fundamental interactions should be built.
The Higgs particle plays a rather enigmatic role. Its phenomenology from radiative corrections and global fits to all observables within the standard model, is not in a satisfactory state. Its model role of providing mass terms for some of the vector bosons and for the fermions of the theory suggests that it be another form of "matter". Models based on noncommutative geometry, in turn, classify the Higgs field in the generalized Yang-Mills connection, besides the gauge bosons, and hence declare it to be part of "radiation". So, what is it?
We worked out some of these themes, by way of construction and by means of instructive examples. We started with a schematic description of Yang-Mills theories including spontaneous symmetry breaking (SSB) within the classical geometric framework, and including matter particles. In a first excursion to quantum field theory we described the stratification of the space of connections and its relevance to anomalies. In order to clarify the phenomenological basis on which Yang-Mills theories of fundamental interactions are built, we reviewed some of the most pertinent phenomenological features of leptons and of quarks. Constructions of the standard model in the framework of noncommutative geometry were briefly summarized. This, in turn lead us to a closer analysis of the mass sector and state mixing phenomena of fermions. The intricacies of quantization were illustrated by a semi-realistic model for massive and massless vector bosons.

Acknowledgements

It is a pleasure to thank the organizers (in alphabetical order) Sergio Adarve, Alexander Cardona, Hernàn Ocampo, Sylvie Paycha, and Andrès Reyes for having organized this wonderful summer school in the old city of Villa de Leyva. Marta Kovacsics deserves special thanks for the excellent administrative coordination and her perfect organization of all aspects of the meeting.
The discussions with the other speakers on various topics of interest, physics, mathematics, life in general, were very stimulating and it was a pleasure to spend quite some time with them.
Last, but certainly not least, I wish to thank the students and other participants at the school for their vivid interest, their questions, and, at times, their patience in more difficult topics (including the art of dancing Salsa).

Exercises

1. What does *compactness* of a Lie group imply for its Killing metric? Why must the structure group of a Yang-Mills theory be compact?

2. Within the framework of a Yang-Mills theory over Minkowski space study the parallel transport of a scalar field by means of the connection.

3. Verify that while the action of a covariant derivative D_A is not *linear,* the action of D_A^2 is.

4. In a unitary representation of a compact simple Lie group one has

   $$\text{tr}\ [U(T_i), U(T_j)] = \kappa\, \delta_{ij}\ .$$

   Show that the constant κ depends on the representation but does not depend on either i or j. Study the examples of spinor and triplet representations of SU(2).

5. Let the structure group be G =SO(3). Construct a Lagrangian for the local gauge theory whose structure group is G, and add a triplet of scalar fields to this Lagrangian.

6. When a YM theory is based on a reductive Lie algebra one says that couplings of matter particles to gauge fields are *universal* in the sense that ratios of physical couplings within a multiplet are fixed. The aim of this exercise is to clarify this statement. Does it hold for $\mathfrak{u}(1)$? So, when are couplings universal?

7. Chirality selection rules: Consider a fermion-fermion vertex coupling to a *scalar* field or to a *vector* field, respectively,

   $$\overline{\psi^{(k)}(x)}\,(a\mathbb{1} + \mathrm{i}b\gamma_5)\,\psi^{(i)}(x)\ ,$$
   $$\overline{\psi^{(k)}(x)}\gamma_\mu\,(a\mathbb{1} + \mathrm{i}b\gamma_5)\,\psi^{(i)}(x)\ .$$

   Work out the chirality selection rules at these vertices.

8. Experiment tells us that the decay $\pi^+ \to e^+\nu_e$ has a probability which is about 10^{-4} smaller than for the decay $\pi^+ \to \mu^+\nu_\mu$, even though the electron is 207 times lighter than the muon. As the available phase space in the electronic decay is much *larger* than in the muonic channel, the decay rate for $\pi^+ \to e^+\nu_e$ should be about five times larger than for $\pi^+ \to \mu^+\nu_\mu$. Making use of the result of the previous exercise can you explain this discrepancy?

9. The decomposition theorem says that any nonsingular matrix M can be written as the product of a (lower) triangular matrix T and a unitary matrix W, $M = TW$. Prove this theorem by induction. Establish its relation to the Schmidt's orthogonalization procedure. Show that the decomposition is unique up to multiplication of W by a diagonal unitary matrix from the left.

10. Show that the Lagrangian (54) of the toy model is BRST-invariant.

Appendix: Proof of Relation (11a)

Let $\Phi = \{\phi^{(i)}\}$ be a multiplet of scalar fields (irreducible representation of SU(2)) such that

$$U^\Phi(T^2)\Phi = t(t+1)\Phi \ , \quad U^\Phi(T_3)\phi^{(i)} = t_3^{(i)}\phi^{(i)} \ , \quad t_3^{(i)} = -t, -t+1, \ldots, t \ .$$

All components of Φ have the same eigenvalue t_0 of the generator T_0 of the U(1) factor, $U^\Phi(T_0)\phi^{(i)} = t_0\phi^{(i)}$. Let $V(\Phi)$ be a quartic potential with a degenerate, absolute minimum at some $\Phi_0 = (\phi_0^{(1)}, \ldots, \phi_0^{(i)}, \ldots)$ which is not identically zero. In the bosonic sector make the ansatz

$$A_\mu^{(0)}(x) = A_\mu^{(\gamma)}(x)\cos\theta_{\rm W} + A_\mu^{(Z)}(x)\sin\theta_{\rm W} \tag{55a}$$

$$bA_\mu^{(3)}(x) = -A_\mu^{(\gamma)}(x)\sin\theta_{\rm W} + A_\mu^{(Z)}(x)\cos\theta_{\rm W} \tag{55b}$$

Here $A_\mu^{(0)}(x)$ and $A_\mu^{(3)}(x)$ are the companions of the generators T_0 and T_3, respectively, while $A_\mu^{(\gamma)}(x)$ and $A_\mu^{(Z)}(x)$ are supposed to become the photon and the Z^0 fields, respectively. The angle $\theta_{\rm W}$ is called the Weinberg angle. In this minimal version it remains a free parameter and has to be determined from experiment. The action of the connection (3a) on the scalar multiplet is

$$U^\Phi(A_\mu)\Phi = {\rm i}q\sum_{k=0}^{3} A_\mu^{(k)}(x)U^\Phi(T_k)\Phi \tag{56}$$

$$= {\rm i}q\left\{\frac{1}{\sqrt{2}}\left[W_\mu^- U^\Phi(T_+) + W_\mu^+ U^\Phi(T_-)\right]\right.$$
$$\left. A_\mu^{(Z)}U^\Phi(T_3\cos\theta_{\rm W} + T_0\sin\theta_{\rm W}) + A_\mu^{(\gamma)}U^\Phi(-T_3\sin\theta_{\rm W} + T_0\cos\theta_{\rm W})\right\}\Phi \ . \tag{57}$$

Here we have replaced T_1 and T_2 by the ladder operators $T_\pm = T_1 \pm {\rm i}T_2$.
Obviously, the factor $(-T_3\sin\theta_{\rm W} + T_0\cos\theta_{\rm W})$ which multiplies the photon field must be proportional to the electric charge operator since only charged particles couple to photons. Now, if Φ_0 has the form $\Phi_0 = (0, 0, \ldots, \phi_0^{(i)} = v \neq 0, 0, \ldots)$ this means that the component $\phi^{(i)}$ of Φ must be electrically neutral. The quantum numbers of the nonvanishing component must be related by the condition

$$t_0^{(i)} = t_3^{(i)}\tan\theta_{\rm W} \ . \tag{58}$$

Another way of expressing this: Φ_0 pertains to the ground state of the theory, i.e. the vacuum. Only electrically neutral fields can develop a nonvanishing vacuum expectation value $v \neq 0$.
Possible mass terms for the vector bosons originate from the scalar product

$$\left(U^\Phi(A_\mu)\Phi_0, U^\Phi(A^\mu)\Phi_0\right) \ . \tag{59}$$

Form here on it is straightforward to compute the term (59) and to isolate the mass terms of $W^\pm$ and Z^0. Making use of the identity $T_+T_- + T_-T_+ = 2(\vec{T}^2 - T_3^2)$ and inserting (58) one obtains

$$\left(U^\Phi(A_\mu)\Phi_0, U^\Phi(A^\mu)\Phi_0\right)$$
$$= q^2v^2\left\{\left[t(t+1) - (t_3^{(i)})^2\right]W_\mu^- W^{+\mu} + \frac{1}{\cos^2\theta_{\rm W}}(t_3^{(i)})^2 A_\mu^{(Z)}A^{(Z)\mu}\right\} \ . \tag{60}$$

One reads off the masses of $W^\pm$ and Z^0 from (6). They are proportional to

$$m_W^2 \propto \frac{1}{2} q^2 v^2 \left[t(t+1) - (t_3^{(i)})^2 \right] \ , \tag{61a}$$

$$m_Z^2 \propto q^2 v^2 \cos^{-2} \theta_{\mathrm{W}} (t_3^{(i)})^2 \ . \tag{61b}$$

As the constant of proportionality is the same for both, the ratio (11a) follows from these equations.
From this derivation one sees very clearly that there is little that restricts the choice of the multiplet for Φ. The only condition is that Φ have one component which is electrically neutral.

References

[1] *Review of Particle Physics,* Phys. Lett. **667** (2008) 1–1340

[2] W. Fetscher, Phys. Lett. **140B** (1984) 117

[3] F. Scheck: *Electroweak and Strong Interactions;*
Springer Verlag Heidelberg, New York, 1996

[4] A. Jodidio et al., Phys. Rev. **D34** (1986) 1967,
A. Jodidio et al., Phys. Rev. **D37** (1988) 237

[5] F.D. Murnaghan, *The Theory of Group Representations,*
Dover, New York 1963,
F.D. Murnaghan, *The Unitary and Rotation Groups,*
Spartan, Washington D.C. 1962

[6] R. Häußling, and F. Scheck; Phys. Lett, **B336** (1994) 477

[7] S. Falk, R. Häußling, and F. Scheck; Phys. Rev **D65** (2002) 093011-1

[8] G.C. Branco, L. Lavoura, and F. Mota; Phys, Rev. **D39** (1989) 3443

[9] R. Häußling, private communication and to be published

[10] B. Kayser, *Neutrino mass, mixing, and flavor change,* in [1]

[11] A. Heil, A. Kersch, N. Papadopoulos, B. Reifenhäuser, and F. Scheck;
Journ. Geom. Phys. **7** (1990) 489,

[12] A. Heil, A. Kersch, N. Papadopoulos, B. Reifenhäuser, F. Scheck, and H. Vogel; Journ. Geom. Phys. **6** (1989) 237

[13] F. Scheck in *Rigorous Methods in Particle Physics,*
S. Ciulli, F. Scheck, W. Thirring (Eds.),
Springer Tracts in Modern Physics **119** (1990) 202

[14] I.M. Singer; Soc. Mathématique de France, Astérisque (1985) 323

[15] A. Connes and J. Lott; Nucl. Phys. **B18** (1990) 29

[16] M. Paschke; Phys. Lett. **B414** (1997) 323

[17] A. Chamseddine and A. Connes; Comm. Math. Phys. **186** (1997) 731

[18] see e.g. A. Connes; *Noncommutative geometry and the standard model with neutrino mixing,* (2006) hep-th/0608226

[19] R. Coquereaux, G. Esposito-Farèse, and F. Scheck;
Int. J. Mod. Phys. **A7** (1992) 6555

[20] R. Coquereaux, R. Häußling, N.A. Papadopoulos, and F. Scheck; Int. J. Mod. Phys. **A7** (1992) 2809

[21] M. Marcu, J. Math. Phys. **21** (1980) 1277

[22] M. Scheunert, *The Theory of Lie-Superalgebras,* Lecture Notes in Mathematics (Springer, 1979)

[23] R. Häußling and F. Scheck; Phys. Lett. **B336** (1994) 477
R. Häußling, M. Paschke, and F. Scheck; Phys. Lett. **B417** (1997) 312

[24] F. Scheck; Phys. Lett. *B284* (1992) 303

[25] *The Gfitter Group,*
H. Flächer, M. Goebel, J. Haller, A. Hoecker, K. Mönig, and J. Stelzer; CERN-OPEN-2008-024, DESY-08-160, arXiv: 0811.0009

[26] G. Scharf; *Quantum Gauge Theories, A True Ghost Story,* John Wiley & Sons, New York 2001

[27] H. Epstein and V. Glaser; Ann. Inst. H. Poincaré **A19** (1973) 211

[28] G. Scharf, *Finite Quantum Electrodynamics, the Causal Approach,* Springer, Berlin, Heidelberg 1995

[29] M. Dütsch, F. Krahe, T. Hurth and G. Scharf, Nuovo Cimento A **106** 1029 (1993); *ibidem* **107** (1994) 375.

[30] M. Dütsch, T. Hurth and G. Scharf,
Nuovo Cimento **A 108** 679 (1995); *ibidem* **A 108** (1995) 737.
M. Dütsch, F. Krahe and G. Scharf, Nuovo Cimento A **106** (1993) 277
M. Dütsch and G. Scharf, Ann. Phys. (Leipzig) **8** (1999) 359
A. Aste, G. Scharf and M. Dütsch, Ann. Phys. (Leipzig) **8** (1999) 389

[31] J.M. Gracia-Bondía, *On the causal gauge principle,* in Proc. of workshop on renormalization, MPIM Bonn; hep-th/0809.0160

[32] M. Dütsch, J. Gracia-Bondía, F. Scheck, and J.C. Varilly, *Quantum Gauge Models without (classical) Higgs Mechanism,* to be published