Theoretical Foundations and Phenomenology of the Fundamental Interactions

Summary of lectures
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Some of the topics covered in these lectures were the following:

Quarks and Leptons, what do we know about them?

- Quantum numbers, selection rules
- Chiral states and selection rules
- Mass sectors and observable state mixing

Geometric construction of non-Abelian gauge theories

- Ideas and expectations
- Differential geometric constructions
- Spontaneous Symmetry Breaking (SSB) and its interpretation

Aspects of the quantization of gauge theories

- Radiative corrections and precision tests
- Anomalies, their rôle and their use in physics

Extensions of the Standard Model

- The Higgs enigma
- Standard model in the framework of noncommutative geometry

Links to gravitation, similarities and differencies

These notes contain only part of the material covered in the lectures. They will be updated and corrected as I continue to work on them.
More on some of the topics dealt with in the lectures can be found, e.g., in the books [1] and [2]. I also refer to my lectures at Villa de Leyva 2009, [3], which can be downloaded from my homepage.

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1 Summary of lecture 1 (2 March 2010)

A general comment: Physics being a *phenomenological* science, rarely, or never makes *absolute* predictions. At best, it establishes *relations* between properties of physical entities or phenomena. Theories are always developed in a specific framework and are adapted to specific *scales.*
Example: 1 Celestial mechanics treats stationary or periodic phenomena, it concerns low velocities (as compared to the speed of light), and weak gravitational fields. Its scale is macroscopic.
Example 2: Quantum mechanics: describes molecules, atoms, nuclei. Velocities are very small, $v \ll c$, space and time scales are 10^{-10} to 10^{-18} m, 10^{-9} to 10^{-18} s, respectively, the energy scales range up to tens of MeV.
In both examples the assumption of a *flat* spacetime is sufficient, i.e. a Euclidean space $\mathbb{R}^4$ with Minkowskian causality structure.

1.1 Charged and neutral leptons

There followed a first short summary of the properties of e^-, $\mu^\pm$, τ-lepton and their neutrino partners ν_e, ν_μ, and ν_τ. More specifically

1. Decay modes of muons, absence of family-number violating processes such as $\mu \to e + \gamma$, $\mu \to e\bar{e}e$ showing that each of the three lepton families carries its own, additively conserved quantum number L_e, L_μ, and L_τ, respectively.

2. Direct vs. indirect measurements of neutrino masses: Direct measurements requiring kinematics *linear* in the mass, are difficult. Best result: $m(\overline{\nu_e}) < 2$ eV from ${}^3\mathrm{H} \to {}^3\mathrm{He} + e^- + \overline{\nu_e}$. Indirect measurements from neutrino oscillations point at much smaller values: One obtains

$$\Delta m^2_{21} = (7.59 \pm 0.20) \times 10^{-5}\ (\mathrm{eV})^2\ ,$$
$$\Delta m^2_{32} = (2.43 \pm 0.13) \times 10^{-3}\ (\mathrm{eV})^2\ .$$

1.2 Chiral states and selection rules

Puzzle: suppression of $\pi^- \to e^-\overline{\nu_e}$ as compared to $\pi^- \to \mu^-\overline{\nu_\mu}$, in spite of the larger phase space available to the former:

$$\frac{\Gamma(\pi^- \to e^-\overline{\nu_e})}{\Gamma(\pi^- \to \mu^-\overline{\nu_\mu})} \simeq \left(\frac{1 - m_e^2/m_\pi^2}{1 - m_\mu^2/m_\pi^2}\right)^2 \frac{m_e^2}{m_\mu^2}\ . \tag{1}$$

The first factor on the r.h.s. gives the value 5.5 and represents the ratio of the available phase space volumes. The second factor whose numerical value is 2.34×10^{-5}, is the real surprise which needs explanation. It will turn out that it is due to a specific helicity selection rule.

Chiral states, definition:

$$\psi_R(x) := \frac{1}{2}\left(\mathbb{1}+\gamma_5\right)\psi(x)\ , \tag{2a}$$

$$\psi_L(x) := \frac{1}{2}\left(\mathbb{1}-\gamma_5\right)\psi(x)\ . \tag{2b}$$

Here $\psi(x)$ is a quantized Dirac field, while

$$P_\pm = \frac{1}{2}\left(\mathbb{1}\pm\gamma_5\right)\ , \quad \text{with } \gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3\ ,$$

are projection operators (please verify).
Onto what do they project? The answer is obtained most easily from the plane wave solutions of the free Dirac equation(s)

$$(\not{p} - m\mathbb{1}_4)u(p) = 0\,, \quad (\not{p} + m\mathbb{1}_4)v(p) = 0\ .$$

In the so-called high-energy (HE) representation one has

$$u(p) = \begin{pmatrix} (1+\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m})\chi \\ (1-\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m})\chi \end{pmatrix}, \quad \text{and} \quad \gamma_5 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}\ .$$

[While $u(p)$ and $v(p)$ are four-component spinors, χ and ϕ are two-component Pauli spinors.] It is then clear that $P_\pm$ project onto the upper or lower components of u. The relevant operator appears to be

$$h := \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\ .$$

If the mass of the fermion vanishes this operator measures the projection of the spin onto the direction of the momentum. Indeed, with $m = 0$ one has $E_p = |\vec{p}|$ and $h = \vec{\sigma}\cdot\vec{p}/|\vec{p}|$. In this case it becomes what is called *helicity*.
Even if the mass is not zero the operator describes *handedness* and is called *chirality operator*. Its interpretation becomes clear when one works out the density matrices for R- or L-handed Dirac fields. For instance, one finds for left-handed fields

$$\varrho_L = P_-(\not{p} - m\not{s})P_+ \tag{3}$$

and from there one calculates the ratio of the probabilities for finding the particle right-handed compared to finding it left-handed

$$\frac{w(h=+1/2)}{w(h=-1/2)} = \frac{E_p - p}{E_p + p} = \frac{1-\beta}{1+\beta} \tag{4}$$

In the case where $E_p \gg m$ this is approximately equal to $1/(4\gamma^2)$ which is to say that the ratio (4) is proportional to m^2/E_p^2. A left-chiral, massive particle has always a certain chance of being found in a right-handed state, even at high energies.

Summary of lecture 2 (4 March 2010)

The analysis of the fields (2a), (2b) shows that

P_+ projects onto *right* chiral *particle* states, and onto *left* chiral *antiparticle* states.

P_- projects onto *left* chiral *particle* states, and onto *right* chiral *antiparticle* states.

From this one draws the following conclusions: Suppose that a vertex coupling to a W-boson has the form $\overline{\psi^{(k)}(x)}\Gamma^\mu\psi^{(i)}(x)$ with

$$\Gamma^\mu = \gamma^\mu(a\mathbb{1} + b\gamma_5) = (a+b)P_-\gamma^\mu P_+ + (a-b)P_+\gamma^\mu P_- \,, \tag{5}$$

with $a, b \in \mathbb{R}$. This is said to be an interaction of V- (vector) and A- (axial vector) type.
In terms of creation and annihilation operators for fermions and antifermions with definite momenta the Dirac field reads

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} \sum_r \int \frac{\mathrm{d}p^3}{2E_p} \left\{ \mathbf{a}^{(r)}(p)u^{(r)}(p)\mathrm{e}^{-\mathrm{i}px} + \mathbf{b}^{(r)\dagger}(p)v^{(r)}(p)\mathrm{e}^{\mathrm{i}px} \right\}$$

while the conjugate field contains the creation operator $\mathbf{a}^{(r)\dagger}(p)$, multiplied by the spinor $\overline{u^{(r)}(p)}$, and the annihilation operator $\mathbf{b}^{(r)}(p)$, multiplied by the spinor $\overline{v^{(r)}(p)}$.
Furthermore, remember two of the elementary Feynman rules of QED: Let p be the four-momentum, r the index that counts the two spin orientations. Then

- An incoming particle comes with $u^{(r)}(p)$,
  an outgoing particle with $\overline{u^{(r)}(p)}$,
- an incoming antiparticle with $\overline{v^{(r)}(p)}$, and
  an outgoing antiparticle with $v^{(r)}(p)$.
- Arrows on the fermion lines are drawn following the flow of *negative* charge. The factors are written down from right to left, following this (these) line(s).

Putting things together we obtain the following selection rule:

$$\boxed{\text{Following this flow, the interaction (5) conserves chirality.}} \tag{6}$$

Remarks:

1. Note that the selection rule (6) holds for arbitrary values of the real parameters a and b, i.e. it applies to an arbitrary mixture of V and A. For instance, the electromagnetic current is of V-type only, hence has $b = 0$. Chirality is conserved, but in this case the two chirality states come in with equal weights.

2. If one considers a vertex of S- (scalar) and P- (pseudoscalar) type instead, $\Gamma = c\mathbb{1} + d\gamma_5$, one sees that the chirality *flips* at every such vertex.

1.3 Example: Two-body decays of charged pions

A remarkable example is provided by the decays $\pi^+ \to e^+\nu_e$ and $\pi^+ \to \mu^+\nu_\mu$. Assume the vertex (5) to have $a = -b = 1$ (the case of V-A interaction), and assume the neutrinos to be massless. In this case the neutrinos are L-chiral, the positron and the μ^+ are R-chiral. The neutrino being massless, its spin is fully aligned in a direction opposite to the momentum. In contrast to this, the R-chirality of the charged partner means that it has a large component where the spin is aligned along the momentum, and a small component where it is aligned antiparallel to the momentum – in the ratio m_f^2/m_π^2, cf. (4), f denoting the charged lepton.
Angular momentum conservation implies that the three-component J_3 of the total angular momentum is conserved. As all partial waves of the plane wave describing the outgoing charged lepton and neutrino have $m_\ell = 0$ (if the relative momentum is taken to be the 3-axis), and as $J_3 = 0$ before the decay, one obtains $m_s^{(f)} + m_s^{(\nu_f)} = 0$ with $f = e$ or $f = \mu$.
Obviously, this selection rule is only obeyed by the *small* component of the charged lepton's chiral field. Thas is to say, if the charged lepton were massless, too, the decay would not take place at all. This clash between the chirality selection rule for V and A interactions and angular momentum conservation explains the smallness of the branching ratio (1).
As an exercise repeat the analogous analysis for the case $a = b = 1$, i.e. for a pure V+A vertex. The conclusions are the same. This shows that *any* mixture of vector and axial vector interaction will lead to the suppression of the decay into the lighter of the charged leptons. This seems reasonable because rates cannot depend on relative signs, i.e. cannot distinguish $V + A$ from $V - A$.

1.4 Muon decay as a model system for charged weak interactions

Muon decay $\mu^- \to e^- \overline{\nu_e} \nu_\mu$ is one of the elementary and most instructive decay processes in weak interactions. Muons are produced abundantly in decays $\pi^- \to \mu^- \overline{\nu_\mu}$, where the newly born muons possess polarization P_μ. The kinematics of muon decay yields the following range of the energy E of the electron

$$m_e \le E \le W = \frac{m_e^2 + m_\mu^2}{2m_\mu} \quad \text{or} \quad x_0 \le x \le 1 \quad \text{where} \quad x := \frac{E}{W}\ , \ x_0 = \frac{m_e}{W}\ .$$

Clearly, unless one is interested in the extreme lower end of the spectrum, one may assume x to scan the interval $0 \le x \le 1$.

Remark: Of course, we know that this decay is due to exchange of a W-boson between the (μ, ν_μ)- and (e, ν_e) vertices. The effect of the mass of the W through its propagator on the muon's decay width is found to be (exercise!)

$$\Gamma = \Gamma(m_W \to \infty) \left\{ 1 + \frac{3}{5} \frac{m_\mu^2}{m_W^2} \right\} .$$

The correction is very tiny. Therefore, the W-propagator may be shrunk to a point, so that the interaction effectively becomes a four-fermion contact interaction.

As, a priori, one does not know the details of the interaction, one tries all admissible coupling terms, each one characterized by a coupling parameter. This analysis, originally due to Louis Michel, yields the following expression for the double-differential decay width

$$\begin{aligned} \frac{\mathrm{d}^2\Gamma}{\mathrm{d}x \mathrm{d}(\cos\theta)} &= A \frac{m_\mu^5 G_F^2}{2^{10}\pi^3 6} x^2 \left\{ \left[6(1-x) + \tfrac{4}{3}\varrho(4x-3) \right] \right. \\ &\quad \left. + P_\mu \xi \cos\theta \left[2(x-1) + \tfrac{4}{3}\delta(3-4x) \right] + \ldots \right\} . \end{aligned} \tag{7}$$

The terms omitted are important only at the low end of the spectrum, or depend on the spin orientation of the electron. The real parameters A, ϱ, ξ, δ, and a few more are linear combinations of squares of coupling constants (not given here). In case of V-A they are $A = 16$, $\varrho = \delta = 3/4$, $\xi = 1$, etc. These values are confirmed in a series of precision experiments (for details see pdg.lbl.gov). Typical results are

$$\begin{aligned} &\varrho = 0.7503 \pm 0.0004 , \quad \delta = 0.7504 \pm 0.0006 , \\ &\xi P_\mu = 1.007 \pm 0.0035 , \quad \eta = 0.001 \pm 0.024 . \end{aligned}$$

Eq. (7) yields interesting information around the upper end of the spectrum. Indeed, one finds from (7)

$$\left(\frac{\mathrm{d}^2\Gamma}{\mathrm{d}x \, \mathrm{d}\cos\theta} \right) \bigg|_{x \to 1} = \frac{m_\mu^5 G_\mathrm{F}^2}{144\pi^3} \varrho \left\{ 1 - P_\mu \frac{\xi\delta}{\varrho} \cos\theta \right\} ,$$

An experiment which measured this quantity found

$$P_\mu \frac{\xi\delta}{\varrho} = 0.9989 \pm 0.0023 .$$

By definition, the longitudinal polarization P_μ cannot be larger than 1. Also, from the definitions in terms of coupling constants, the absolute value of the combination $(\xi\delta/\varrho)$ cannot exceed the value 1. One concludes that both P_μ and the combination $\xi\delta/\varrho$ must each lie very close to 1. The result for the former has an immediate consequence for the chirality of the $\overline{\nu_\mu}$ of the antineutrino emitted in pion decay, by conservation of angular momentum, the experimental result given above implies

$$1 - 2|h(\overline{\nu}_\mu)| < 0.0032 \quad \text{at} \quad 90\% \text{ C.L.} \,.$$

The *signs* of $h(\overline{\nu_\mu})$ and $h(\nu_\mu)$, which are opposite of each other, are known from another experiment. Therefore, the result is convertible to the information

$$h(\overline{\nu_\mu}) = +\frac{1}{2} \quad \text{and} \quad h(\nu_\mu) = -\frac{1}{2}\,, \tag{8}$$

within very small error bars. This is, by far, the most accurate determination of a neutrino helicity. Note that in case of spin-1/2 it is customary to define the helicity twice these values, i.e. $h(\overline{\nu_f}) = +1$ and $h(\nu_f) = -1$.

In summary:

> Charged Current (CC) weak interactions couple to L-chiral fields only. Neutrinos (antineutrinos), as long as they are considered massless, couple only by left-helicity (right-helicity).
> There are Neutral Current (NC) weak interactions as well to which we will turn later.

2 Mass sectors of leptons and quarks

Summary of lectures 3 and 4 (9 and 11 March 2010)

2.1 Mass matrices and mixing

With $\Psi(x)$ a multi-component fermion field containing all leptons (or quarks) of equal charge, the Lagrange density will have the form

$$\mathcal{L} = \overline{\Psi(x)}\,(\mathrm{i}\gamma^\mu\partial_\mu - M)\,\Psi(x) + \text{h.c.}\,, \tag{9}$$

where M is a matrix which need neither be hermitean nor diagonal. It will not be diagonal if Ψ refers to a basis of *weak interaction eigenstates* and if these do not coincide with the *mass eigenstates.*
The mass matrix is a Lorentz-scalar. With regard to the space of Dirac spinors, the decomposition in terms of L- and R-chiral fields shows that the mass term is of the form $-\mathcal{L}_{\text{mass}} = \overline{\Psi_R} M \Psi_L + \text{h.c.}$. Indeed, one has

$$\overline{\Psi_{R/L}} = (P_\pm\Psi)^\dagger\gamma^0 = \Psi^\dagger P_\pm\gamma^0 = \Psi^\dagger\gamma^0 P_\mp = \overline{\Psi}P_\mp\,, \text{hence}$$
$$\overline{\Psi}\Psi = (\overline{\Psi_R} + \overline{\Psi_L})(\Psi_R + \Psi_L) = \overline{\Psi}(P_- + P_+)(P_+ + P_-)\Psi\,,$$

of which only the cross terms survive. Note that the CC weak interactions couple to L-fields only and are blind to the R-fields.

Introduce the following notation: For *weak interaction states* let the Dirac spinors for particles with the higher value of the electric charge be denoted generically by $u^{(i)}(x)$, for particles with the lower electric charge by $d^{(i)}(x)$, the superscript $i = 1, 2, 3$ counts the families or generations. So, in the case of leptons, the particles with electric charge $Q = 0$ are the neutrinos, while the ones with $Q = -1$ are the charged leptons e^-, μ^-, and τ^-. In the case of quarks $u^{(i)}$ stands for the three *up*-quarks with charge $+4/3$, $d^{(i)}$ stands for the three down quarks with charge $-1/3$, as they appear in CC weak interactions.
More explicitly, the mass sector in the Lagrangian has the form

$$-\mathcal{L}_{\text{mass}} = \frac{1}{2}\left\{\sum_{i,k=1}^{3} \overline{u_L^{(i)}} M_{ik}^{(u)} u_R^{(k)} + \sum_{i,k=1}^{3} \overline{d_L^{(i)}} M_{ik}^{(d)} d_R^{(k)}\right\} + \text{ h.c.} . \tag{10}$$

The two mass matrices that appear here, are diagonalized by bi-unitary transformations,

$$U_L^{(u)} M^{(u)} U_R^{(u)} = D^{(u)} \quad \text{and} \quad U_L^{(d)} M^{(d)} U_R^{(d)} = D^{(d)} \,, \text{ with}$$
$$D^{(u)} = \text{diag}\,(m_1^{(u)}, m_2^{(u)}, m_3^{(u)}) \quad \text{and} \quad D^{(d)} = \text{diag}\,(m_1^{(d)}, m_2^{(d)}, m_3^{(d)})\,.$$

These formulae depend on four independent unitary matrices. It is not difficult to verify that the determination of these unitaries can be reduced to diagonalization of *hermitean* matrices, (dropping the superscripts (u) or (d))

$$U_R \left(M^\dagger M\right) U_R^\dagger = D^2 = U_L \left(M M^\dagger\right) U_L^\dagger \,.$$

The corresponding eigenstates are the *mass eigenstates*. We denote the L-chiral eigenstates by $t_L^{(i)}$ and $b_L^{(i)}$ in the two charge sectors, respectively. They are

$$t_L^{(i)} = \sum_k \left(U_L^{(u)}\right)_{ik} u_L^{(k)} \,, \quad b_L^{(i)} = \sum_k \left(U_L^{(d)}\right)_{ik} d_L^{(k)} \,. \tag{11}$$

Note that the CC weak interaction vertices are proportional to

$$\overline{u_L^{(i)}} \gamma^\mu (\mathbb{1} - \gamma_5) d_L^{(k)} \,.$$

Therefore, when expressed in terms of mass eigenstates, these vertices are characterized by the mixing matrix

$$U_L^{(u)}\, U_L^{(d)\dagger} =: V \,. \tag{12}$$

In the case of quarks $V \equiv V_{\text{CKM}}$ is the Cabibbo-Kobayashi-Maskawa mixing matrix. In the case of leptons it has no generally accepted name.

2.2 Some comments

1. In the case of quarks the mass eigenstates are the states which are characterized by quantum numbers such as strangeness, etc., all of which are conserved in strange interactions but not in CC weak interactions. The mixing matrix may hence be visualized as a kind of rotation between states participating in strong interactions and states participating in CC weak interactions.

2. Number of observable parameters in V: Being unitary, V a priori depends on nine real parameters. As five relative phases (six minus one overall phase) can be absorbed in the initial and final states, there remain four observable variables. With three generations one takes them to be three real mixing angles $\theta_1, \theta_2, \theta_3$ and one (real) phase angle ϕ.

3. It is instructive to consider the most general transformation which leaves V invariant:
   Let U, $W^{(u)}$, and $W^{(d)}$ be *arbitrary* unitary matrices. Then the simultaneous transformations of the mass matrices

   $$U^\dagger\, M^{(u)}\, W^{(u)} \quad \text{and} \quad U^\dagger\, M^{(d)}\, W^{(d)} \tag{13}$$

   leave V, eq. (12), unchanged (please verify!).

4. While the route from a set of given matrices to the mixing matrix is clear, it is of interest to determine the set of all mass matrices that are compatible with a set of given data, i.e. with the three masses m_1, m_2, m_3 and the four observables derived from the mixing matrix V. (For details and references see FS's talk at Villa de Leyva summer school 2010, accessible through web page.) This may be useful for model builders.

5. In relation to the previous comment it useful to know that by a suitable choice of basis states one can transform *any* set of mass matrices to the form

   $$\hat{M} = \begin{pmatrix} 0 & * & 0 \\ * & 0 & * \\ 0 & * & * \end{pmatrix}$$

   where the asterisks indicate nonvanishing entries. In the literature on the subject this is called the NNI representation (for "nearest neighbour interactions"). Although this is a useful representation in analyses of mass matrices, the name is misleading because it suggests a specific pattern in the interaction responsible for the mass sectors, although there is none.

6. It is obvious that by making use of the freedom (13) the mixing can be shifted from one charge sector to the other, or may even be split

between the two charge sectors. Note that by convention one takes the *down*-quarks to be the mixed states in the quark sector, and the neutrinos to be the mixed states in the leptonic sector.

3 Geometric Construction of Non-Abelian Gauge Theories

Summary of lectures 5 and 6 (16 and 18 March)

3.1 Ideas and expectations

Matter: Leptons are classified as described above, i.e. doublets for L-fields, and singlets for R-fields

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L , e^-_R \quad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L , \mu^-_R \quad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L , \tau^-_R , \tag{14a}$$

With respect to the U(1) factor in the group $U(2) \simeq U_Y(1) \times SU_I(2)$ the doublets have $Y = -1$, the singlets have $Y = -2$, so that the electric charge is given by $Q = I_3 + \frac{1}{2}Y$.
Quarks appear in three copies of the reducible multiplet containing a $SU(2)_I$ doublet and two singlets,

$$\Psi^{(1)} = \begin{pmatrix} \begin{pmatrix} u_L \\ d_L \end{pmatrix} \\ u_R \\ d_R \end{pmatrix} \quad \Psi^{(2)} = \begin{pmatrix} \begin{pmatrix} c_L \\ s_L \end{pmatrix} \\ c_R \\ s_R \end{pmatrix} \quad \Psi^{(3)} = \begin{pmatrix} \begin{pmatrix} t_L \\ b_L \end{pmatrix} \\ t_R \\ b_R \end{pmatrix} \tag{14b}$$

Up-quarks have electric charge $Q = +\frac{2}{3}$, *down*-quarks have $Q = -\frac{1}{3}$. The weak hypercharges of the doublets are $Y = \frac{1}{3}$, u_R, c_R, and t_R have $Y = \frac{4}{3}$, while d_R, s_R, and b_R have $Y = -\frac{2}{3}$. One verifies their electric charges by the formula $Q = I_3 + \frac{1}{2}Y$.

Aims: Want to incorporate: The neutral-current (NC) phenomenology of electromagnetic and weak interactions, the maximal violation of parity in CC interactions, the phenomenology and structure of QCD. (Regarding the latter subject, we repeated the motivation for the colour degrees of freedom as obtained from the constituent quark model and the spin-statistics relation for quarks.)
Following the model of quantized Maxwell theory (QED) as a successful, predictive renormalizable quantum field theory, quest for a renormalizable unified theory.

Table 1: Quantum numbers of leptons (L=L_e+L_μ+L_τ)

	Q	L	L_e	L_μ	L_τ	I_3	Y
$(\nu_e)_L$	0	1	1	0	0	$+\frac{1}{2}$	-1
$(e^-)_L$	-1	1	1	0	0	$-\frac{1}{2}$	-1
$(e^-)_R$	-1	1	1	0	0	0	-2
$(\nu_\mu)_L$	0	1	0	1	0	$+\frac{1}{2}$	-1
$(\mu^-)_L$	-1	1	0	1	0	$-\frac{1}{2}$	-1
$(\mu^-)_R$	-1	1	0	1	0	0	-2
$(\nu_\tau)_L$	0	1	0	0	1	$+\frac{1}{2}$	-1
$(\tau^-)_L$	-1	1	0	0	1	$-\frac{1}{2}$	-1
$(\tau^-)_R$	-1	1	0	0	1	0	-2

The two tables summarize the quantum numbers of leptons and quarks as discussed above. The last two columns show the weak isospin and weak hypercharge assignments which are relevant for the electroweak (minimal) standard model. Note that the electric charge is $Q = I_3 + \frac{1}{2}Y$.

Table 2: Quantum numbers of quarks

	Q	B	C	S	To	Bo	I_3	Y
u_L	$\frac{2}{3}$	$\frac{1}{3}$	0	0	0	0	$+\frac{1}{2}$	$+\frac{1}{3}$
d_L	$-\frac{1}{3}$	$\frac{1}{3}$	0	0	0	0	$-\frac{1}{2}$	$+\frac{1}{3}$
u_R	$\frac{2}{3}$	$\frac{1}{3}$	0	0	0	0	0	$+\frac{4}{3}$
d_R	$-\frac{1}{3}$	$\frac{1}{3}$	0	0	0	0	0	$-\frac{2}{3}$
c_L	$\frac{2}{3}$	$\frac{1}{3}$	1	0	0	0	$+\frac{1}{2}$	$\frac{1}{3}$
s_L	$-\frac{1}{3}$	$\frac{1}{3}$	0	-1	0	0	$-\frac{1}{2}$	$\frac{1}{3}$
c_R	$\frac{2}{3}$	$\frac{1}{3}$	1	0	0	0	0	$+\frac{4}{3}$
s_R	$-\frac{1}{3}$	$\frac{1}{3}$	0	-1	0	0	0	$-\frac{2}{3}$
t_L	$\frac{2}{3}$	$\frac{1}{3}$	0	0	1	0	$+\frac{1}{2}$	$\frac{1}{3}$
b_L	$-\frac{1}{3}$	$\frac{1}{3}$	0	0	0	-1	$-\frac{1}{2}$	$\frac{1}{3}$
t_R	$\frac{2}{3}$	$\frac{1}{3}$	0	0	1	0	0	$+\frac{4}{3}$
b_R	$-\frac{1}{3}$	$\frac{1}{3}$	0	0	0	-1	0	$-\frac{2}{3}$

3.2 Geometry of Maxwell theory

The relevant fields are the four-potential $A^\mu(x)$ and the field strength tensor field $F^{\mu\nu}(x)$, which, when expressed in a given frame of reference, are

$$A^\mu(x) = \Big(\Phi(t,\vec{x}), \vec{A}(t,\vec{x})\Big)^T \ , \ F^{\mu\nu}(x) = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} . \quad (15)$$

The homogeneous and inhomogeneous Maxwell equations (in vacuum) read, respectively,

$$\partial^\lambda F^{\mu\nu}(x) + (\text{cyclic permutations of } \lambda, \mu, \nu) = 0\,, \quad (16a)$$
$$\partial_\mu F^{\mu\nu}(x) = 0\,. \quad (16b)$$

Gauge invariance: If $A^\mu(x) \mapsto A'^\mu(x) = A^\mu(x) - \partial^\mu\chi(x)$, with $\chi(x)$ a Lorentz scalar function (which is at least C^2), the field strength tensor and hence the electric and magnetic fields remain unchanged.

Example: Consider the Schrödinger equation for a charged particle subject to external electric and magnetic fields, the Hamiltonian being

$$H = \frac{1}{2m}\left[\vec{p} - \frac{e}{c}\vec{A}\right]^2 + e\Phi\,.$$

If one performs a time and space dependent phase transformation on the wave function by means of an arbitrary smooth function $\alpha(t,\vec{x})$

$$\psi(t,\vec{x}) \longmapsto \psi'(t,\vec{x}) = \mathrm{e}^{\mathrm{i}\alpha(t,\vec{x})}\psi(t,\vec{x}) \quad (17a)$$

then the equation of motion remains unchanged if one performs simultaneously a gauge transformation of the fields by means of the gauge function

$$\chi(t,\vec{x}) = \frac{\hbar c}{e}\alpha(t,\vec{x})\,. \quad (17b)$$

(Work this out as an exercise.) This reationship was found by V. Fock in 1926. It shows very clearly that in the case of Maxwell theory the relevant symmetry group is $G =$U(1).

Let M denote Minkowski space, i.e. the space $\mathbb{R}^4$ equipped with the metric $g = \text{diag }(1,-1,-1,-1)$, let $\Lambda^* = \oplus_k \Lambda^k$ be the space of exterior forms on M, and $d : \Lambda^k \to \Lambda^{k+1}$ the exterior derivative.
Define the following one- and two-forms, respectively,

$$\omega_A := A_\mu(x)\mathrm{d}x^\mu\,, \quad \omega_F := \sum_{\mu<\nu} F_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu\,. \quad (18)$$

One verifies easily that $\omega_F = \mathrm{d}\omega_A$ is identical with the well-known expression $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$ which guarantees the validity of the homogeneous equations (16a). A gauge transformation takes the form

$$\omega_A \longmapsto \omega_{A'} = \omega_A + \mathrm{d}\chi(x) \quad \Longrightarrow \omega_{F'} = \omega_F \, .$$

The covariant derivative which acts on matter fields (bosonic or fermionic) is

$$D_A := \mathrm{d} + \mathrm{i}\frac{q}{\hbar c}\omega_A = \mathrm{i}\left(-\mathrm{i}\mathrm{d} + \frac{q}{\hbar c}\omega_A\right) \tag{19}$$

where the second form reminds one of $(\vec{p} - \frac{q}{c}\vec{A})$ with $\vec{p} = -\mathrm{i}\hbar\nabla$.
Of course, one chooses units such that $\hbar = 1$ and $c = 1$. It is convenient to absorb the factor i and the coupling constant q (which is the electric charge in the case of Maxwell theory) in the definition of the exterior forms (18),

$$A := \mathrm{i}\frac{q}{\hbar c}\omega_A \, , \quad F := \mathrm{i}\frac{q}{\hbar c}\omega_F \, . \tag{20}$$

Then the above definitions are rewritten as follows

$$\begin{aligned} &D_A = \mathrm{d} + A \, , \quad F = \mathrm{d}A \, , \text{and one calculates} \\ &D_A^2 = (\mathrm{d} + A) \circ (\mathrm{d} + A) = \mathrm{d} \circ \mathrm{d} + \mathrm{d}A + A\mathrm{d} \\ &= (\mathrm{d}A) - A\mathrm{d} + A\mathrm{d} = (\mathrm{d}A) = F \, , \end{aligned}$$

where $\mathrm{d}^2 = 0$ and the graded Leibniz rule for the exterior derivative were used. The covariant derivative acts on bosonic or fermionic fields, $D_A\Phi$ or $D_A\Psi$, scalar products such as $(D_A\Psi, D_A\Psi)$ will be invariant under gauge transformations $A \mapsto A' = A + \mathrm{d}\Lambda$ provided the action of G on the field is

$$\Psi(x) \mapsto \Psi'(x) = g(x)\Psi(x) \quad \text{with} \quad g(x) = \mathrm{e}^{\Lambda} \, .$$

3.3 Compact non-Abelian groups and their algebras

One starts from the *structure group* G and its Lie algebra $\mathfrak{g} \equiv \mathrm{Lie(G)}$. Here G is a compact Lie group, simple or semi-simple.The generators of $\mathfrak{g}$ are denoted by T_i, and they fulfill the commutators (summation over repeated indices is implied)

$$[T_i, T_j] = \mathrm{i}C_{ij}^k T_k \, . \tag{21}$$

From the Jacobi identity for the generators one deduces the following relation for the structure constants

$$C_{ij}^m C_{mk}^l + C_{jk}^m C_{mi}^l + C_{ki}^m C_{mj}^l = 0 \, .$$

Among the irreducible, unitary representations most important for our construction is the *adjoint representation*, where the generators are represented by the matrices

$$U_{lm}^{(\mathrm{ad})}(T_k) = -\mathrm{i}C_{kl}^m \, .$$

(Exercise: Verify that these matrices do indeed fulfill the commutators (21).)
The *Killing metric* is defined by

$$g_{ij} := \mathrm{tr}\ \left(U^{(\mathrm{ad}}(T_i)U^{(\mathrm{ad}}(T_j)\right) = -C^p_{iq}C^q_{jp} \tag{22}$$

Finally, by the definition $C_{ijk} := C^p_{ij}g_{pk}$ one obtains the structure constants in a form where they are antisymmetric in all three indices. (Verify this by means of the identity derived from the Jacobi identity!)
An important property which is used in the construction of local gauge theory is the following: By suitable linear combination the generators can be chosen such that

$$\mathrm{tr}\ \left(U(T_i)U(T_j)\right) = \kappa\delta_{ij}\,, \tag{23}$$

where the real constant κ depends on the representation but is independent of the generators T_i. (We gave a proof of this fact.)
The example of SU(2):

$$T_i = \frac{\sigma_i}{2}\,,\ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

yields the fundamental representation for which $\kappa^{(\mathrm{fund})} = 1/2$.
The adjoint representation is given by

$$U^{(\mathrm{ad})}(T_1) = -\mathrm{i}\{\varepsilon_{1lm}\} = -\mathrm{i}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix},$$

$$U^{(\mathrm{ad})}(T_2) = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}, \quad U^{(\mathrm{ad})}(T_3) = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

from which one calculates $\kappa^{(\mathrm{ad})} = 2$.

3.4 Globally and locally invariant Lagrange densities

With G denoting the structure group, construct the principal fibre bundle (M, G) with base manifold $M = \mathbb{R}^4$ and typical fiber G. This construction also defines the infinite dimensional *gauge group* $\mathcal{G}$ (loosely speaking: to each point $x \in M$ a copy ${}^xG$ of the structure group is attached). Define the Lie algebra-valued one-form, with $N = \dim\ \mathrm{Lie}(G)$,

$$A := \mathrm{i}q \sum_{k=1}^{N} A^{(k)}T_k \quad \text{with} \quad A^k = A^{(k)}_\mu(x)\mathrm{d}x^\mu, k = 1, 2, \ldots, N, \tag{24}$$

In any reducible or irreducible representation of G spanned by the components of some (bosonic or fermionic) matter field Φ, A will be realized by a matrix $U(A) = \mathrm{i}q\sum_{k=1}^{N} A^{(k)}U(T_k)$. For example, if Φ spans an m-dimensional representation, these will be $m \times m$ matrices.

The one-form A is called *connection form*. Indeed, it describes parallel transport of the field Φ from x to $x + \mathrm{d}x$,

$$\phi_i^{(x+\mathrm{d}x)} = \sum_{j=1}^{m} \{\delta_{ij} - U_{ij}(A)\} \, \phi_j^{(x)} ,$$

provided A transforms under a gauge transformation $g(x) \in \mathcal{G}$ as follows

$$A \longmapsto A' = gAg^{-1} + g(\mathrm{d}g^{-1}) . \tag{25}$$

Schematically this is shown as follows. A natural requirement is that parallel transport should commute with the action of a gauge transformation, that is to say, one should have

$$g(x + \mathrm{d}x)(\mathbb{1} - A) = (\mathbb{1} - A')g(x) . \tag{26}$$

Upon Taylor expansion to first order, $g(x+\mathrm{d}x) \simeq g(x)+\mathrm{d}g(x)$, eq. (26) gives

$$\mathrm{d}g - gA = -A'g \quad \text{or} \quad A' = gAg^{-1} - (\mathrm{d}g)g^{-1} .$$

Note that $\mathrm{d}(gg^{-1}) = 0 = (\mathrm{d}g)g^{-1} + g(\mathrm{d}g^{-1})$ which proves (25).
Next one defines the *covariant derivative*

$$D_A := \mathrm{d} + A , \tag{27}$$

in complete analogy to Maxwell theory, as well as the *curvature two-form*

$$F := (\mathrm{d}A) + A \wedge A . \tag{28}$$

Note that in the case of Maxwell theory the structure group was $G =$U(1), and the second term on the right-hand side in (28) vanished. Furthermore,

$$D_A^2 = (\mathrm{d} + A) \circ (\mathrm{d} + A) = \mathrm{d}A + A\mathrm{d} + A \wedge A = (\mathrm{d}A) + A \wedge A = F .$$

We showed that under a gauge transformation D_A transforms by "conjugation", $D_A \mapsto D_{A'} = gAg^{-1}$. As $F = D_A^2$, this is also true for the curvature; $F \mapsto F' = gFg^{-1}$. This will be important when we will have to construct invariants under the gauge group $\mathcal{G}$.
The term in F which is new as compared to Maxwell theory is

$$\begin{aligned}
A \wedge A &= -q^2 \sum_{k,l=1}^{N} T_k T_l \sum_{\sigma,\tau=0}^{3} A_\sigma^{(k)}(x) A_\tau^{(l)}(x) \, \mathrm{d}x^\sigma \wedge \mathrm{d}x^\tau \\
&= -q^2 \sum_{k,l=1}^{N} \left[T_k, T_l\right] \sum_{\mu<\nu} A_\mu^{(k)}(x) A_\nu^{(l)}(x) \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\
&= -\mathrm{i}q^2 \sum_{k,l,m=1}^{N} C_{klm} T_m \sum_{\mu<\nu} A_\mu^{(k)}(x) A_\nu^{(l)}(x) \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu .
\end{aligned}$$

Obviously, F is a Lie algebra-valued two-form. Therefore, it is natural to write it as a linear combination $F = \mathrm{i}q\sum_k T_k \sum_{\mu<\nu} F^{(k)}_{\mu\nu}(x)\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ in terms of the generators. The coefficients in this expression are the following functions

$$F^{(k)}_{\mu\nu}(x) = \partial_\mu A^{(k)}_\nu - \partial_\nu A^{(k)}_\mu - q \sum_{m,n=1}^{N} C_{kmn} A^{(m)}_\mu(x) A^{(n)}_\nu(x)\,. \tag{29}$$

The first term on the right-hand side is the same as for the Maxwell field. The second term is new and is typical for non-Abelian theories.
The *Lagrange density* for a gauge theory with matter must be *globally* invariant (i.e. with respect to the structure group G) and *locally* invariant (i.e.with respect to the gauge group $\mathcal{G}$). Its generic structure will be

$$\mathcal{L} = \mathcal{L}_{\mathrm{YM}} + \mathcal{L}_\Phi + \mathcal{L}_\Psi\,, \quad \text{where}$$

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4q^2\kappa^{(\mathrm{ad})}}\mathrm{tr}\ \left(F_{\mu\nu}F^{\mu\nu}\right)\,, \tag{30a}$$

$$\mathcal{L}_\Phi = \tfrac{1}{2}\left\{(D_A\Phi, D_A\Phi) - \mu^2\,(\Phi,\Phi)\right\} - W(\Phi)\,, \tag{30b}$$

$$\mathcal{L}_\Psi = \tfrac{\mathrm{i}}{2}\left(\overline{\Psi}, \gamma^\alpha \overleftrightarrow{D}_\alpha\, \Psi\right) - \left(\overline{\Psi}, (M + g\Phi)\Psi\right)\,. \tag{30c}$$

The "left-right" notation of the covariant derivative in (30c) is a short-hand for $\overline{\Psi}(D_\alpha\Psi) - (D_\alpha\overline{\Psi})\Psi$. The "bracket" notation $(\cdots,\cdots)$ is somewhat symbolic. It is meant to indicate that the two objects on the right and on the left should be coupled to a *scalar* with respect to G (global invariance!). For example, if G =SO(3) and $\vec{\Phi} = \{\phi_1, \phi_2, \phi_3\}$ is a triplet, (Φ,Φ) stands for the scalar product $(\vec{\Phi}\cdot\vec{\Phi})$. Likewise, if G =SU(3) and Ψ is an octet **8**, then $\overline{\Psi}$ (which is the conjugate octet) and Ψ must be coupled according to the Clebsch-Gordan series $\mathbf{8}\otimes\mathbf{8} \to \mathbf{1}$. In this way all terms in $\mathcal{L}$ are G-invariant.

3.5 The U(2) model of electroweak interactions

Summary of lectures 6 and 7 (23 and 25 March)
The generators of the Lie algebra of U(2) are taken to be

$$T_0 = \mathbb{1}_2 \equiv \sigma_0, \quad T_i = \tfrac{1}{2}\sigma_i\,.$$

It is convenient to replace T_1 and T_2 by the ladder operators $T_\pm = T_1 \pm \mathrm{i}T_2$. This implies that the corresponding gauge fields are replaced by

$$W^\pm_\mu(x) = \tfrac{1}{\sqrt{2}}\{A^{(1)}_\mu(x) \pm \mathrm{i}A^{(2)}_\mu(x)\}\,.$$

and will serve to describe the two charge states of the W-boson. The Yang-Mills Lagrangian (30a) contains the terms

$$T_1 A^{(1)}_\mu(x) + T_2 A^{(2)}_\mu(x) = \tfrac{1}{\sqrt{2}}\left\{T_- W^{(+)}_\mu + T_+ W^{(-)}_\mu\right\}\,. \tag{31}$$

It also contains neutral terms $A_\mu^{(0)}$ pertaining to the U(1) factor in G, and $A_\mu^{(3)}$ corresponding to the 3-component of SU(2). The physical photon and Z^0 are expected to be linear combinations thereof, viz.

$$A_\mu^{(\gamma)}(x) = A_\mu^{(0)}(x)\cos\theta_\mathrm{W} - A_\mu^{(3)}(x)\sin\theta_\mathrm{W}\,, \tag{32a}$$

$$A_\mu^{(Z)}(x) = A_\mu^{(0)}(x)\sin\theta_\mathrm{W} + A_\mu^{(3)}(x)\cos\theta_\mathrm{W}\,, \tag{32b}$$

with θ_W an angle remaining undetermined at this stage, called *Weinberg angle.* The neutral sector of $\mathcal{L}_\mathrm{YM}$ then becomes

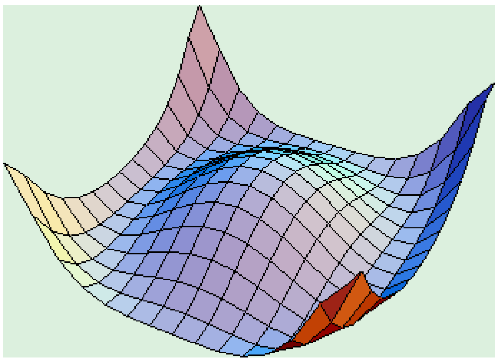
$$A_\mu^{(Z)}(x)\,(T_3\cos\theta_\mathrm{W} + T_0\sin\theta_\mathrm{W}) + A_\mu^{(\gamma)}(x)\,(-T_3\sin\theta_\mathrm{W} + T_0\cos\theta_\mathrm{W})\ . \tag{32c}$$

So far, all vector bosons of the theory are massless. (Adding ad-hoc terms quadratic in $W_\mu^\pm$ and $A_\mu^{(Z)}$ does not help because such terms are not gauge invariant.) They have a chance of obtaining masses – without destroying gauge invariance – only via $\mathcal{L}_\Phi$, eq. (30b), in the case where in the lowest (vacuum) state Φ has a nonvanishing value Φ_0. Then, indeed, the covariant derivative terms in (30b) will yield mass terms for some or all of the vector bosons. However, if this happens, we must make sure that the photon remains massless! This can only be achieved if the eigenvalues of T_0 and T_3 in the state Φ_0 are such that in the above decomposition the coefficient multiplying $A_\mu^{(\gamma)}(x)$ vanishes.

Thus, assume that the potential $W(\Phi)$ in (30b) is such that it has a degenerate minimum at

$$\Phi_0 = (0, 0, \dots, 0, \phi_0^{(i)}, 0, \dots, 0)^T$$

with $\phi_0^{(i)} \equiv v \neq 0$. The *dynamic* Higgs field will then be $\Theta = \Phi - \Phi_0$.



A somewhat symbolic sketch of a Higgs potential which shows a degenerate minimum $\phi_0^{(i)} \neq 0$

The whole multiplet Φ, or Θ, is characterized by a quantum number t (corresponding to the eigenvalue $t(t+1)$ of the operator $\vec{T}^2$ of SU(2)). Its components are distinguished by the eigenvalues $t_3^{(k)} = -t, -t+1, \dots, +t$ of T_3. So assume $t_3^{(i)}$ to be the eigenvalue pertaining to $\phi_0^{(i)}$, and $t_0^{(i)}$ its eigenvalue of the generator T_0. The photon will remain massless provided these quantum numbers are related by

$$t_0^{(i)} = t_3^{(i)}\tan\theta_\mathrm{W}\,.$$

Regarding the $W^\pm$ and the Z^0, the terms in Φ_0 that follow from (30b) are then

$$
\begin{aligned}
&\big(U^{(\Phi)}(A_\mu)\Phi_0, U^{(\Phi)}(A_\mu)\Phi_0\big) \\
&= q^2\Big\{\tfrac{1}{2}\left(\Phi_0, U^{(\Phi)}(T_+T_- + T_-T_+)\Phi_0\right) W_\mu^{(-)}(x)W_\mu^{(+)}(x) \\
&+ \Big(\Phi_0, U^{(\Phi)}(T_3\cos\theta_{\mathrm{W}} + t_3^{(i)}\sin\theta_{\mathrm{W}}\tan\theta_{\mathrm{W}})^2\Phi_0\Big) A_\mu^{(Z)}(x)A^{(Z)\mu}(x)\Big\} .
\end{aligned}
$$

Of course, the operator T_3 in the third line can be replaced by its eigenvalue $t_3^{(i)}$ and, similarly, the sum of products of ladder operators in the second line can be replaced by diagonal operators. Indeed, taking account of

$$
\begin{aligned}
&\tfrac{1}{2}\left(T_+T_- + T_-T_+\right) = \vec{T}^2 - T_3^2 \quad \text{and of} \\
&\cos\theta_{\mathrm{W}} + \sin\theta_{\mathrm{W}}\tan\theta_{\mathrm{W}} = \frac{1}{\cos\theta_{\mathrm{W}}} ,
\end{aligned}
$$

one obtains genuine mass terms for the $W^\pm$ and for the Z^0 as follows

$$
\begin{aligned}
&\big(U^{(\Phi)}(A_\mu)\Phi_0, U^{(\Phi)}(A_\mu)\Phi_0\big) \\
&= q^2v^2\left\{\left[t(t+1) - t_3^{(i)\,2}\right]W_\mu^{(-)}(x)W_\mu^{(+)}(x) + \frac{t_3^{(i)\,2}}{\cos^2\theta_{\mathrm{W}}}A_\mu^{(Z)}(x)A^{(Z)\mu}(x)\right\} .
\end{aligned}
$$

From his expression one deduces that

$$
m_W^2 \propto \tfrac{1}{2}q^2v^2\left[t(t+1) - t_3^{(i)\,2}\right] \quad \text{and} \quad m_Z^2 \propto q^2v^2\frac{1}{\cos^2\theta_{\mathrm{W}}}t_3^{(i)\,2} ,
$$

from which follows a relation which contains only experimental quantities on one side, only the quantum numbers of $\phi_0^{(i)}$ on the other,

$$
\frac{m_W^2}{m_Z^2\cos^2\theta_{\mathrm{W}}} = \frac{t(t+1) - t_3^{(i)\,2}}{2t_3^{(i)\,2}} . \tag{33}
$$

The left-hand side of (33) is known to be 1 within a very small experimental error bar. One verifies that the right-hand side takes that value for $t = \frac{1}{2}$ and $t_3^{(i)} = \pm\frac{1}{2}$. Here a number of remarks should be made.

1. The component $\phi^{(i)}$ of Φ which develops this vacuum expectation value $\phi_0^{(i)} = v$ must be uncharged. This is achieved by the relation $t_0^{(i)} = t_3^{(i)}\tan\theta_{\mathrm{W}}$ obtained above. Turning back to the photon coupling in (32c) one sees that it can be written as

   $$
   (-q\sin\theta_{\mathrm{W}})\left[T_3 - T_0\frac{\cos\theta_{\mathrm{W}}}{\sin\theta_{\mathrm{W}}}\right] .
   $$

   The factor in front is the elementary charge $-q\sin\theta_{\mathrm{W}} = e$. If one wants to recover the formula $Q = T_3 + \frac{1}{2}Y$, then the U(1) generator must be rescaled by defining

   $$
   Y = -2\,\frac{\cos\theta_{\mathrm{W}}}{\sin\theta_{\mathrm{W}}}\,T_0 . \tag{34}
   $$

This is always possible because this factor is a U(1) symmetry for which the eigenvalues have no scale.

2. Note that the original symmetry is broken from U(2)$\sim$U$_Y$(1)$\times$SU$_I$(2) to the residual symmetry U$_{\text{Maxwell}}$(1) of Maxwell theory. The Lagrange density still has the full symmetry, the ground state has only the residual symmetry.

3. This reduction or "hiding" of the symmetry has a nice geometric interpretation in the framework of Goldstone's theorem.

4. Returning to the explicit expression (29) one realizes that the vector boson sector (30a) of the Lagrange density contains cubic and quartic couplings among vector bosons. These are genuine interaction terms which are new and which are characteristic for non-Abelian theories. For example, these terms fix the anomalous magnetic moment of the W-boson through the interaction term with the photon.
   Joining the SU$_c$(3) of QCD to the gauge group and thereby adding the eight massless gluons to the sector of vector bosons, the Lagrangian $\mathcal{L}_{\text{YM}}$ contains three- and four-gluon couplings. These were tested successfully in jet physics.

5. A U(1) gauge theory such as Maxwell's theory does not have such self-interactions of the (only) vector boson. Indeed, light-by-light scattering is possible only in fourth order of QED, via a box diagram involving four internal fermion lines (Delbrück scattering).

3.6 SU(3)$_c$ and Quantum ChromoDynamics (QCD)

Building stones of QCD are the structure group $G=$SU(3)$_c$, the colour group, together with the corresponding gauge group, and $\Psi : q^{A,\alpha}(x)$, the quark fields. Here $A = u, d, c, s, t, b$ is the flavour index, $\alpha = 1, 2, 3$ (or *red*, *white*, and *blue*) is the colour index. The index α corresponds to the triplet representation **3** of SU(3)$_c$, three-quark states are classified by the singlet of the Clebsch-Gordan series

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1}_{\text{a}} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10}_{\text{s}} \, .$$

As the singlet representation is totally antisymmetric in its three factors the conflict of the quark model with the spin-statistics relation is resolved. Indeed, with $\varepsilon_{\alpha\beta\gamma}$ being an invariant tensor in SU(3), a baryon state is constructed by taking

$$\sum_{\alpha,\beta,\gamma=1}^{3} \varepsilon_{\alpha\beta\gamma} q^{A,\alpha} q^{B,\beta} q^{C,\gamma} \, ,$$

where the flavour quantum numbers A, B, C must be composed such as to yield the correct quantum numbers of the baryons. So, even if the flavour state is totally symmetric, and if the orbital wave function is symmetric, too, the total wave function will be antisymmetric by virtue of the colour wave function. As an example reconsider the state of a Δ^{++} with spin projection $m_s = 3/2$:

$$(u_\uparrow u_\uparrow u_\uparrow) \quad \text{is replaced by} \quad \sum_{\alpha,\beta,\gamma=1}^{3} \varepsilon_{\alpha\beta\gamma}(u_\uparrow^\alpha u_\uparrow^\beta u_\uparrow^\gamma)\,.$$

The group G =SU(3) is generated by the eight generators T_k of its Lie algebra

$$g \in \text{SU}(3)\ :\ g = \exp\{ih\} \quad \text{with} \quad h = \sum_{k=1}^{8} \alpha_k T_k \ \text{(hermitean)}.$$

The dimension of the Lie algebra is 8, the rank of the group is 2. The commutators are

$$[T_i, T_k] = \mathrm{i}\sum_{l=1}^{8} f_{ikl} T_l\,, \quad i,k = 1,2,\dots,8\,, \tag{35}$$

the structure constants (in totally antisymmetric form) being as follows,

Table 3: Structure constants for SU(3)

ikl	123	147	156	246	257	345	367	458	678
f_{ikl}	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$

All other constants are either obtained from these by the antisymmetry in (i,k,l), or they vanish, $f_{mnp} = 0$ in case (m,n,p) is not a permutation of the (i,k,l) listed in the table. In the defining (or fundamental) representation the generators can be expressed in terms of the Gell-Mann matrices λ_k, $T_k = \lambda_k/2$, where

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ \lambda_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \ \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \ \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

In the *fundamental representation* **3** one has

$$\text{tr}\ (T_i T_k) = \frac{1}{4}\text{tr}\ (\lambda_i \lambda_k) = \frac{1}{2}\delta_{ik}\,, \quad \text{hence} \quad \kappa^{(\text{fund})} = \frac{1}{2}\,.$$

The *adjoint representation* is defined by $(T_k)_{mn} = -\mathrm{i} f_{kmn}$, the structure constants being given by Table 3. The relevant κ parameter is calculated by

$$\mathrm{tr}\ \left(T_i^2\right) = -\sum_{m,n=1}^{8} f_{imn} f_{inm} = +\sum_{m,n=1}^{8} f_{imn}^2 \quad \text{giving} \quad \kappa^{(\mathrm{ad})} = \frac{3}{2}\,.$$

The QCD-Lagrangian is then constructed as formulated in eqs. (29) and (30a). It describes 8 massless vector bosons, the *gluons* which are interpreted as the carriers of strong interactions. However, while the Lagrange function for QCD is simple, its ground state is a highly correlated state (*confinement!*).

4 The minimal standard model and beyond

4.1 Some comments

1. The standard model works all too well: The model as such, as well as the radiative corrections it allows to calculate, were brillantly confirmed by the precision data obtained at LEP and at the Tevatron. Global fits to all data (masses of W and Z, their widths Γ_W and Γ_Z, various forward-backward asymmetries in the neighbourhood of the Z-pole in e^+e^- scattering, etc.) are of high quality and leave no room for deviations from the minimal version of the standard model.

   As we explained in detail, the standard model is a parametrization and summary of the empirical knowledge of its building blocks and their interactions at low energies (maximal parity violation in CC-interactions, neutral current phenomenology, neutrinos being pure L-fields, etc.).

2. The model is free of anomalies. Although it exhibits a U(1) anomaly (which manifests itself in the succesful prediction of the $\pi^0 \to \gamma\gamma$ decay rate), the contribution of the three lepton families cancels the contribution of the three families of (coloured) quarks (see below, Sect. 4.3). As a consequence, the model can be quantized in a consistent manner so that radiative corrections are finite and well-defined.

3. The model, unfortunately, contains too many free parameters (some 24 in all!). An example is provided by the fermionic mass sector: As we showed in the lectures there can be no genuine fermion masses because they cannot be reconciled with gauge symmetry. It is not possible to form a G-scalar from a L-chiral field, which is a member of a doublet, and a R-chiral field which is a singlet with respect to $\mathrm{SU}(2)_I$. If instead the fermions are given their masses by the Higgs mechanism, then the Yukawa couplings of the Higgs field to each lepton or quark must be chosen proportional to the masses one wishes to obtain.
   As an example consider the L-doublet $L^{(f)} = (\nu_f, f^-)^T$ with weak hypercharge $Y = -1$, and the R-chiral field $(f^-)_R$ with $Y = -2$, as well

as the Higgs doublet Φ with $Y = 1$. As $\overline{L^{(f)}}$ then has $Y = +1$, a Yukawa coupling of the type

$$\mathcal{L}_{\text{Yukawa}} = g^{(f)} \left\{ \left(\overline{L^{(f)}} \Phi \right) f_R^- + \text{ h.c.} \right\} \tag{36}$$

is (globally) invariant (which is sufficient here): The additive conservation of weak hypercharge is respected. The two $\mathrm{SU}(2)_{\mathrm{I}}$ doublets are coupled to a scalar. The price one has to pay is that the Yukawa coupling constant must be adjusted such as to give the correct fermion mass. Typically, with Φ replaced by Φ_0 and with $(\Phi_0, \Phi_0) = v^2$ one obtains

$$g^{(f)} = \frac{m_f}{v} \quad \text{with} \quad v \simeq 250 \text{ GeV}.$$

This "explains" why the couplings to the physical Higgs particle are expected to be very small and, consequently, why the production rates of Higgs particles are low.

So one may claim that the standard model does not really unify the interactions. In essence, it contains too little structure: While it fixes the sector of vector bosons to a large extent (self-couplings of non-Abelian vector fields), it leaves much freedom in the choice of multiplets for the matter particles, leptons and quarks. Also, the Higgs sector remains somewhat enigmatic.

4. There are many attempts to endow the model with more structure by embedding it in larger groups (grand unified theories), or by introducing space-time supersymmetry, or in more exotic versions such as technicolour, etc. They all have in common that even more parameters are introduced which are not determined, not less.

5. Matters that we did not have time to discuss in more detail:

   - *BRST symmetry and its implications.* The symmetry operation discovered by Becchi, Rouet, Stora, and Tyutin is an elegant method to implement gauge invariance. This operation introduces a further, graded algebraic and geometric structure which, when combined with the requirements of locality and causality à la Epstein and Glaser, leads to another, deeper perspective of gauge theories.
   - *Perturbative and nonperturbative regimes of gauge theories.*

4.2 Anomalies in gauge theories: Generalities

Geometrically speaking, anomalies are topological obstructions in the construction of the action for a Yang-Mills theory. If a gauge theory exhibits anomalies it cannot be quantized properly, that is to say, the theory will not make sense at the quantum level, as a renormalizable quantum field theory. Qualitatively speaking, the reason for this complication is the following.

The space $\mathcal{A}$ of all connections is an affine space. This would be a fairly simple and perfectly manageable space were it not for gauge invariance of non-Abelian theories. Gauge symmetry means that there are *classes* of gauge-equivalent connections $[A]$ and, in essence, that the full gauge group $\mathcal{G}$ should be divided out. However, the space $\mathcal{A}/\mathcal{G}$ has a complicated structure and this division must be done more carefully, in a series of steps, while following the fate of the action functional along this procedure. A good way of studying this process is to make use of the stratification of the space $\mathcal{A}$ by the gauge group, the strata being defined by equivalence classes of gauge-equivalent connections.

For that purpose one studies the stratification of $\mathfrak{A}$ by the action of the gauge group, so that it is decomposed as follows

$$\mathfrak{A} = \mathfrak{A}^{(J_0)} \cup \mathfrak{A}^{(J_1)} \cup \cdots \mathfrak{A}^{(J_k)} \; , \tag{37a}$$

where the individual stratum is characterized by the stability group $\mathfrak{G}_A$ of its elements A being conjugate to J_i, $i = 0, \ldots, k$. That is to say

$$\mathfrak{A}^{(J_i)} = \left\{ A \in \mathfrak{A} | \, \mathfrak{G}_A = \psi J_i \psi^{-1} \; , \psi \in \mathfrak{G} \right\} \; . \tag{37b}$$

Here J_i is isomorphic to a subgroup of the structure group G. The number of strata is countable. The main stratum (index J_0) is characterized by J_0 isomorphic to the center C of G. The stratification (37a) is unique and, in fact, natural. The formal definition

$$\mathcal{M}_i = \mathfrak{A}^{(J_i)} / \mathfrak{G}$$

yields an orbit bundle decomposition which bears some similarity to the compactification procedure in Kaluza-Klein theories. The spaces $\mathcal{M}_i$ are parts of the space of physical, gauge-inequivalent connections.

Physics comes in via an action functional $S(A, \bar{\psi}, \psi)$ which is a classical functional that is strictly gauge invariant. At the quantum level the central quantity is the generating function

$$Z(A) = \int [\mathcal{D}\chi][\mathcal{D}\bar{\chi}] \exp\{-(S + \bar{\chi}\, \not{\partial}_A \, \chi)\} \tag{38}$$

obtained while integrating the fermionic degrees of freedom. There are two possibilities. Either $Z(A)$ is strictly invariant under a gauge transformation ψ, $Z(\psi A) = Z(A)$, or it is equivariant but not strictly invariant, $Z(\psi A) = \varrho^{-1}(A, \psi) Z(A)$, where ϱ is the action of the gauge transformation. In the first case one can safely divide by the gauge group to obtain a perfectly acceptable functional. The theory has no anomaly and can be reduced to the space $\mathcal{M}$ of gauge-inequivalent connections. In the second case, in contrast, there must be anomalies, the reduction is not possible. Therefore, in the geometric framework anomalies show up as obstructions to the reduction procedure which are due to quantization.

One well-known ansatz is to make use of what is called the *pointed gauge group* $\mathfrak{G}^*$. This is the subgroup of $\mathfrak{G}$ which is the stability group of an arbitrary but fixed point p_0 of the principal fibre bundle $\mathcal{P}(M, G)$,

$$\mathfrak{G}^* = \mathfrak{G}_{p_0} = \{\psi \in \mathfrak{G} | \, \psi(p_0) = p_0\} \ . \tag{39}$$

In other terms the pointed gauge group acts like the identity in the fibre over p_0. Singer showed that the action of $\mathfrak{G}^*$ on $\mathfrak{A}$ is free [4]. Therefore,

$$\begin{array}{ccc} \mathfrak{G}^* & \longrightarrow & \mathfrak{A} \\ & & \downarrow \\ & & \mathcal{M}^* = \mathfrak{A}/\mathfrak{G}^* \end{array}$$

is a principal fibre bundle. The functional $Z(A)$ is a trivial section

$$Z \, : \, \mathfrak{A} \longrightarrow \text{Det} \, := \mathfrak{A} \times \mathbb{C} \tag{40a}$$

in the determinant bundle. If one divides by the pointed gauge group one obtains the reduced section

$$Z^* \, : \, \mathfrak{A}/\mathfrak{G}^* \longrightarrow (\mathfrak{A} \times \mathbb{C}) \, / \mathfrak{G}^* =: \text{Det}^{\,*} \, . \tag{40b}$$

If $Z(A)$ is strictly invariant then the action of $\mathfrak{G}^*$ on $\mathbb{C}$ is trivial so that (40b) reduces to

$$Z^* \, : \, \mathcal{M}^* \longrightarrow \mathcal{M}^* \times \mathbb{C} \, , \quad \mathcal{M}^* = \mathfrak{A}/\mathfrak{G}^* \, . \tag{40c}$$

In turn, if $Z(A)$ is equivariant but not strictly invariant then the action of the pointed gauge group on $\mathbb{C}$ is not trivial. In this situation Det^* has a twist, the integration over $[\mathcal{D}A]$ is not possible. Geometrically speaking there is a topological anomaly.
Even if no such anomaly is encountered, the story is not finished. There remains the "division" by the remainder $\mathfrak{G}/\mathfrak{G}^*$ which is isomorphic to the structure group G. This last step is particularly important because it is the structure group which defines the conserved charges of the theory. Again, if Z^* is strictly invariant the final division poses no problem. If it is not but is (only) equivariant one obtains

$$Z^{**} \, : \, \mathcal{M} \longrightarrow (\mathcal{M}^* \times \mathbb{C}) \, /G =: \text{Det}^{**} \, , \tag{41}$$

the functional Z^{**} is nontrivial, and one has found an anomaly.
In summary, by following this geometrical method one identifies all topological as well as possible global, nonperturbative anomalies. More on this can be found e.g. in [4], and in [5] – [7].

4.3 Anomalies: An example

The electroweak sector of the standard model is based on $\text{U}(2) \simeq \text{U}(1)_Y \times \text{SU}(2)_I$. The SU(2) factor does not yield an anomaly. However, the U(1) part contains

a chiral anomaly that was the first that was discovered in the 1960-ties by Adler, Bell, and Jackiw. This example is instructive for two reasons. First, it shows that anomalies may be there but they may cancel due to the specific fermion content of the model. Second, the axial anomaly, when confined to the contributions of quarks, may be useful in predicting the decay rate for $\pi^0 \to \gamma\gamma$ in a quantitative manner.
Define the axial vector current J_A^μ and the axial density J_A of some fermion field(s) as follows

$$J_A^\mu(x) := \overline{\psi(x)}\gamma^\mu\gamma_5\psi(x)\,, \tag{42a}$$

$$J_A(x) := \overline{\psi(x)}\gamma_5\psi(x)\,. \tag{42b}$$

Here ψ may be a single fermion field $\psi \equiv \psi^{(f)}$, or a multi-component Dirac field $\psi(x) \equiv \Psi(x)$ which contains different species of fermions that are distinguished by their quantum numbers.
The neutral pion π^0 has spin/parity 0^-, so that the divergence $\partial_\mu J_A^\mu$ of the current (42a) is a suitable interpolating field for the pion. The process $\pi^0 \to \gamma\gamma$ involves three vertices:

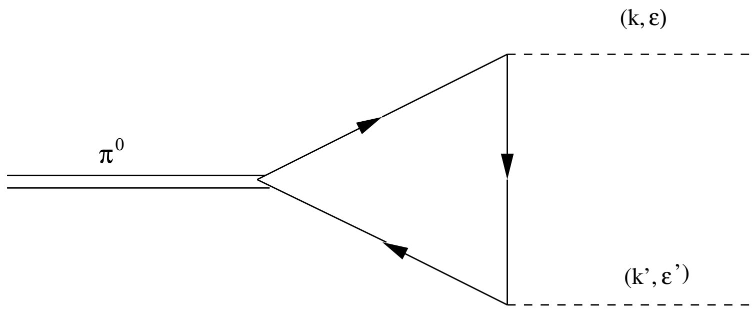


Figure 1: Triangle graph illustrating π^0 decay via the axial anomaly

Two *vector* vertices proportional to γ^α and γ^β at which the two photons (k, ε) and (k', ε') are created, respectively, (the first symbol denoting the four-momentum, the second the polarization), and the pseudoscalar $\partial_\mu J_A^\mu$ describing the dissociation of the π^0 into two quarks. From general considerations, regarding spin and parity selection rules, the decay amplitude must have the form

$$T(\pi^0 \to \gamma\gamma) = F\ \varepsilon_{\alpha\beta\sigma\tau}k^\alpha k'^\beta \varepsilon^\sigma \varepsilon'^\tau\,, \tag{43a}$$

where F is a Lorentz-scalar form factor whose absolute square determines the decay width, or inverse lifetime,

$$\tau^{-1} = \frac{m_\pi^3}{64\pi}|F|^2\,. \tag{43b}$$

Naïvely one would expect that F tend to zero as $m_\pi^2 \to 0$. "Naïve" here means that while π^0 is described by $\partial_\mu J_A^\mu$, this divergence is calculated by making use of the equations of motion that follow from the Lagrange density,

$$\partial_\mu J^\mu_A = 2\mathrm{i} m J_A(x)\,, \tag{44a}$$

where m is a quark mass. This calculation is purely formal because one did not check whether the relevant Feynman diagram is convergent or not. In fact, the triangle diagram is found to be (linearly) divergent and must be regularized in order to make sense. If one does so (as was first done by St. Adler) one finds instead of (44a)

$$\partial_\mu J^\mu_A = 2\mathrm{i} m J_A(x) + \frac{\alpha}{4\pi}\varepsilon_{\alpha\beta\sigma\tau}F^{\alpha\beta}(x)F^{\sigma\tau}(x)\,. \tag{44b}$$

The second term in (44b) is said to be the *axial anomaly.* We return to it and to its use for π^0-decay further below. For the moment we consider the more general case of a triangle graph with two vector currents and one axial-vector current, but allow both leptons and quarks to circulate along the loop. One then shows that if there were only the leptons the triangle anomaly would be proportional to

$$S_{\text{leptons}} = \sum_{e,\mu,\tau} \text{tr}\,\left(T_3^2 Y\right) = 3\cdot\frac{1}{4}(-2) = -\frac{3}{2}\,. \tag{45a}$$

Obviously, only the doublets contribute which are purely left-fields. In turn, if there were only quarks in nature, i.e. the three generations, and each in three differents colours, then the anomaly would be proportional to

$$S_{\text{quarks}} = 3_{\text{c}} \sum_{q-\text{doublets}} \text{tr}\,\left(T_3^2 Y\right) = 3_{\text{c}}\left(3_{\text{F}}\cdot\frac{1}{4}\cdot\frac{2}{3}\right) = +\frac{3}{2}\,. \tag{45b}$$

Within the framework of the standard model the anomaly vanishes because the contribution (45a) of the leptons cancels against the contribution (45b) of the quarks. Note that the colour factor 3_{c} is essential for this cancellation to happen!
Returning to π^0-decay we note that here only quarks can contribute, the π^0 being a $(u\overline{u} + d\overline{d})$ state in the constituent quark model. The axial current is then

$$A^\alpha(x) = \sum_q c_q\,\overline{\psi_q(x)}\gamma^\alpha\gamma_5\psi_q(x)\,,\ \text{with } c_u = \tfrac{1}{2}\,,\ c_d = -\tfrac{1}{2}\,. \tag{46}$$

The anomaly is proportional to $S = \sum_q c_q Q^2(q)$ where $Q(q)$ is the electric charge, $+2/3$ for the *up*-, $-1/3$ for the *down*-quark. The decay amplitude in (43a) is calculated to be

$$F = -\frac{\alpha}{\pi}2S\,\frac{g_{\pi NN}}{m_N F_A}\,, \tag{47}$$

where α is the fine-structure constant, $g_{\pi NN} \simeq 13.6$ is the pion-nucleon coupling constant (determined from pion-nucleon scattering), $F_A = -1.27$ is the ratio of axial-vector to vector coupling obtained, e.g., from neutron β-decay, $n \to p + e^- + \overline{\nu_e}$, and $m_N = 939$ MeV is the mass of the nucleon. If one inserts these numbers into the expression (43b) one obtains the prediction

$$\tau = 7.7 \times 10^{-17}\ \mathrm{s}/(4|S|^2)\,. \tag{48}$$

The factor S is calculated to be

$$S = 3_{\mathrm{c}} \left[\frac{1}{2} \left(\frac{2}{3} \right)^2 - \frac{1}{2} \left(\frac{1}{3} \right)^2 \right] = \frac{1}{2}\,,$$

so that $4|S|^2 = 1$. If this is so the result (48) is found to be in very good agreement with the experimental result

$$\tau_{\mathrm{exp}} = (8.4 \pm 0.6) \times 10^{-17}\ \mathrm{s}. \tag{49}$$

This result is remarkable for several reasons. In principle, the occurence of anomalies is *bad* because they may ruin renormalizability. The standard model of electroweak interactions avoids this catastrophy by a conspiracy of quarks and leptons whose contributions cancel. This is one of the few instances where the symmetry between the lepton families and the quark families is essential. On the other hand, the axial anomaly is *good* and welcome because it is measurable in π^0-decay. In particular, the lifetime comes out correctly only because there is the colour factor. Without this factor one would miss the experimental value by a factor of 9.

4.4 The standard model within noncommutative geometry

There are various approaches to the standard model which use a more general geometric setting within what is called *noncommutative geometry.* I gave a summary of the basic ideas in the Villa de Leyva lectures to which I refer. Here, as an alternative, I give a short description within a framework that may be called *algebraic Yang-Mills-Higgs theories.*
Schematically, all extensions of the geometric framework described above are based on some exterior algebra $\Omega^*(\mathcal{A})$ over the algebra $\mathcal{A}$, equipped with a product denoted by $\bullet$ and an exterior derivative denoted by d,

$$(\Omega^*(\mathcal{A}), \bullet, \mathrm{d}) \tag{50}$$

Of course, the algebra, the product and the exterior derivative are to be specified. Furthermore, a connection A has to be chosen within the framework (50). For example, in the so-called commutative case, i.e. in the case of gauge theories as described in previous lectures, the algebra is the (commutative) algebra of smooth functions on Minkowski space M^4, the product is the wedge product, and the exterior derivative is the Cartan derivative $\mathrm{d_C}$. The exterior algebra is the tensor product of the exterior algebra on M^4 and the Lie algebra of the structure group, viz.

$$\mathcal{A} = C^\infty(M^4),\ \bullet = \wedge,\ d = d_{\mathrm{C}},\ \Omega^* = \Lambda^*(M^4) \otimes \mathrm{Lie}(G)\,. \tag{51}$$

The connection is an element of Ω^*, more precisely, a Lie-algebra valued one-form.
Before moving on, a few remarks should be made:

(i) When developing a generalization making use of noncommutative geometry, the commutative case (51) must be contained as a limit. It must be possible to "switch off" the noncommutative structure and to return to ordinary Yang-Mills-Higgs theory.

(ii) In different models the connection often is the same. But this is not so for the product and for the exterior derivative. Therefore, the curvature $F = \mathrm{d}A + A \bullet A$ or $F = \nabla^2$ will be different in different models.

(iii) In models such as the Mainz-Marseille model the Dirac operator is a derived quantity. In models based on the Connes-Lott approach, the Dirac operator plays the role of the driving agent for the geometry.

(iv) Models based on a *spectral action* refer to the analogy to classical General Relativity (Chamseddine-Connes).

In the framework of algebraic YMH theories there are various gradings:

(A) Exterior forms in $\Lambda^*(M^4)$ carry what one calls a *form grade.* Scalar fields are functions and, hence, zero-forms. The connection is a one-form, while curvature is a two-form. There is indeed a $\mathbb{Z}_2$-grading of $\Lambda^*(M^4)$ given by its $\mathbb{N}$-grading modulo 2.

(B) Left- and right-chiral fermion fields also provide a natural grading. Indeed, the spinor representations of SL(2,$\mathbb{C}$) span a vector space $C = C^{(L)} \oplus C^{(R)}$. Furthermore, the L- and R-fields are representations of $\mathrm{SU}(2)_I$, i.e. live in vector spaces $X^{(L)}$ and $X^{(R)}$, respectively. Thus, left- and right-chiral fields are defined on the spaces

$$V^{(0)} := C^{(L)} \otimes X^{(L)} \quad \text{and} \quad V^{(1)} := C^{(R)} \otimes X^{(R)} . \tag{52}$$

which may be interpreted as *even* and *odd*, respectively. This grading is called the *chirality grading.*

(C) Finally, the Clifford algebra $\mathcal{C}\ell(M^4)$ carries a natural $\mathbb{Z}_2$-grading,

$$\Gamma^{(0)} = \{\mathbb{1}, \gamma^\mu\gamma^\nu, \mathrm{i}\gamma_5\} \ , \quad \Gamma^{(1)} = \{\gamma^\mu, (\mathrm{i}\gamma_5)\gamma^\mu\} \ . \tag{53}$$

The elements of $\Gamma^{(0)}$ are caracterized by the property that they commute with γ_5, while the elements of $\Gamma^{(1)}$ *anti*commute with γ_5.

An algebraic YMH model such as the Mainz-Marseille model is based on three hypotheses:
Hypothesis I: The gradings (B) and (C) are identified. This is rather natural

because the Clifford algebra $\mathcal{C}\ell(M^4)$ as a vector space is isomorphic to the exterior algebra $\Lambda^*(M^4)$. For instance, one has the correspondence

$$\Phi(x) \leftrightarrow \Phi\mathbb{1}\,, \quad A_\mu(x)\,\mathrm{d}x^\mu \leftrightarrow A_\mu(x)\gamma^\mu\,, \quad F_{\mu\nu}(x)\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \leftrightarrow F_{\mu\nu}\gamma^\mu\gamma^\nu\,.$$

Hypothesis II: Assume the Lie algebra Lie(G) to be embedded in the minimal super Lie algebra whose even part coincides with Lie(G). The $\mathbb{Z}_2$-grading of the latter is identical with the one in (A), (B), and (C). Then one obtains

$$\mathrm{SU}(2)\times\mathrm{U}(1) \subset \mathrm{SU}(2|1) = \left\{M \in M_2(\mathbb{C}) | M^\dagger = -M, \mathrm{Str}\; M = 0\right\}. \quad (54)$$

The even part of SU(2|1) is generated by the operators T_i, $i = 1, 2, 3$, of weak isospin, and by T_0 or Y for weak hypercharge. In addition there are four odd generators Ω_a and Ω'_b, $a, b = 1, 2$, all of which can be written as 3×3-matrices

$$\left(\begin{array}{cc|c} * & * & * \\ * & * & * \\ \hline * & * & * \end{array}\right) \equiv \left(\begin{array}{c|c} \mathbf{A} & \mathbf{C} \\ \hline \mathbf{D} & \mathbf{B} \end{array}\right),$$

whose blocks along the main diagonal belong to the even part of the algebra, while the off-diagonal rectangles belong to its odd part. All generators have vanishing supertrace. Among them the property Str $(Y) = \mathrm{tr}\;(A) - \mathrm{tr}\;(B) = 0$ guarantees that there are no anomalies.
The connection which is a one-form with values in SU(2|1) is

$$A = \mathrm{i}\left\{a\vec{T}\cdot\vec{W} + \frac{1}{2}bYW^{(8)}\right\} + \mathrm{i}\frac{c}{\mu}\left\{\Phi^{(0)}\Omega'_- + \Phi^{(+)}\Omega'_+ + \mathrm{h.c.}\right\} \quad (55)$$

So far, the algebra and the connection are fixed. The product $\bullet$ is easily seen to be the wedge product for the forms (which are entries of the matrices in SU(2|1)). Regarding *even* and *odd* matrices (denoted by (0) and (1), respectively,) the product or commutators must be

$$M \bullet N = M^{(0)}N^{(0)} + M^{(0)}N^{(1)} + M^{(1)}N^{(0)} + \mathrm{i}M^{(1)}N^{(1)}\,, \quad (56a)$$
$$[M, N]_g = \left[M^{(0)}, N^{(0)}\right]_- + \left[M^{(0)}, N^{(1)}\right]_- + \left[M^{(1)}, N^{(0)}\right]_- + \mathrm{i}\left[M^{(1)}, N^{(1)}\right]_+\,. \quad (56b)$$

Hypothesis III: The exterior derivative is defined such that the doubly graded structure of the algebra is preserved. This requirement can be achieved by supplementing the Cartan derivative d_C by a *matrix derivative* d_M which is defined to be the graded commutator with a specific odd element η of SU(2|1),

$$\mathrm{d} = \mathrm{d}_\mathrm{C} + \mathrm{d}_M \quad \text{with} \quad \mathrm{d}_M = [\eta, \cdot] \quad (\eta \text{ odd}). \quad (57)$$

The curvature which is obtained from the connection (55) and from the derivative (57) via the structure equation

$$F = \mathrm{d}A + \frac{1}{2}[A, A]_g + \mathrm{const.}$$

is found to be the Yang-Mills curvature plus Higgs-like terms and a constant background field. Indeed, there is a constant connection $A_0 = -\eta$ which is invariant under all constant gauge transformations

$$A \longmapsto A' = A + \mathrm{d}\varepsilon + [A, \varepsilon]_g \,, \quad \varepsilon = \varepsilon^{(0)} + \varepsilon^{(1)} \,. \tag{58}$$

The corresponding curvature is found to be

$$F_0 = \mathrm{i}\left(T_3 + \frac{1}{2}Y\right) \,. \tag{59}$$

This is seen to be the charge operator which appears here as a kind of invariant background field. The residual symmetry of the theory must be $\mathrm{U}(1)_{\mathrm{e.m.}}$ which is the stability group of the charge operator (59). This explains at once why this model yields the Higgs potential and spontaneous symmetry breaking (in the correct phase of the scalar field). For further details and the extension to quark and lepton fields I refer to the Villa de Leyva lectures and to the original publications.

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